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Guaranteed in-control performance of the EWMA chart for monitoring the mean

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Abstract
Research on the performance evaluation and the design of the Phase II EWMA control chart for monitoring the mean, when parameters are estimated, have mainly focused on the marginal in-control average run-length (ARL_{IN}). Recent research has highlighted the high variability in the in-control performance of these charts. This has led to the recommendation of studying of the conditional in-control average run-length ($CARL_{IN}$) distribution. We study the performance and the design of the Phase II EWMA chart for the mean, using the $CARL_{IN}$ distribution and the exceedance probability criterion (*EPC*). The $CARL_{IN}$ distribution is approximated by the Markov Chain method and Monte Carlo simulations. Our results show that in-order to design charts that guarantee a specified *EPC*, more Phase I data are needed than previously recommended in the literature. A method for adjusting the Phase II EWMA control chart limits, to achieve a specified *EPC*, for the available amount of data at hand, is presented. This method does not involve bootstrapping and produces results that are about the same as some existing results. Tables and graphs of the adjusted constants are provided. An in-control and out-of-control performance evaluation of the adjusted limits EWMA chart is presented. Results show that, for moderate to large shifts, the performance of the adjusted limits EWMA chart is quite satisfactory. For small shifts, an in-control and out-of-control performance tradeoff can be made to improve performance.

KEYWORDS
bootstrap, conditional average run-length, exceedance probability criterion, exponentially weighted moving average chart, Markov chain, unconditional and conditional perspectives

1 | INTRODUCTION

Jones et al[1] studied the conditional and the unconditional run-length distribution of EWMA chart with estimated parameters in both the in-control (*IC)* and the out-of-control (*OOC)* cases. Based on the percentage increase in the false alarm rate (*FAR*), they concluded that when parameters are estimated and the smoothing constant (λ) is small, larger Phase I sample sizes are needed, to design charts with acceptable *FAR* performance. However, their study did not take into account the random variability of the *FAR*, the so-called "practitioner to practitioner" variability, which is inherent to parameter estimation. Motivated by this, Saleh et al[2] examined the $CARL_{IN}$ distribution of the EWMA chart as a function of the number of Phase I subgroups (m), subgroup size (n), and λ. Based on the standard deviation ($SDCARL_{IN}$) of $CARL_{IN}$, they concluded, contrary to Jones et al,[1] that, much larger Phase I sample sizes are required to design Phase II EWMA charts with larger λ than with smaller λ.

In his seminal work on prospective application of the Phase II $\overline{X}$ chart, Chakraborti[3] was among the first group of authors to highlight the variation present in the conditional run-length distribution and hence the importance of examining the practitioner to practitioner variability via the conditional run-length distribution. He emphasized how the conditional false alarm rate (*CFAR)* behaves as a random variable when parameters are estimated and used to construct Phase II charts. Inspired by this, for the Phase II S and S^2 charts, Epprecht et al[4] examined the *CFAR* distribution as a function of the Phase I sample size mn. They then made recommendations about the minimum size of the Phase I sample, which is required, to guarantee, with a high probability $1 - p$, that the *CFAR* will not exceed some specified nominal *CFAR* value (denoted $CFAR_0$). This is the exceedance probability criterion (*EPC*) introduced by Albers et al[5] and Gandy and Kvaloy[6] which sets an upper prediction bound to *CFAR*. In the same spirit, we examine the $CARL_{IN}$ distribution of the EWMA chart as a function of λ, m, and $n = 5$ and set a lower prediction bound to $CARL_{IN}$. We then make recommendations about the value of m, which is required, to guarantee, with a specified high probability 1-p, that the $CARL_{IN}$ will exceed a nominally specified value, denoted ARL_0. This approach has been recommended in the recent literature as the $CARL_{IN}$ (also the *CFAR*) is a random variables with high variability which is the cause of practitioner to practitioner variation. Our results reveal that in order for the EWMA chart to meet the *EPC* specification, even more Phase I data are needed than was previously recommended by Saleh et al[2] and Jones et al.[1] Moreover, consistently with Jones et al[1] but contrary to Saleh et al,[2] it will be seen that small values of λ require larger Phase I sample sizes than large values of λ.

However, in practice, it may be difficult and expensive to get such huge amounts of Phase I data. Hence, control limits are adjusted as a function of the amount of data available at hand. Jones[7] adjusted the control limits of the Phase II EWMA chart to achieve a certain nominally specified marginal or unconditional in-control *ARL* (ARL_0). This is the unconditional approach. Values of the charting constant (L) were given graphically for different values of ARL_0, m, n, and choice of λ ranging from 0.02 to 1. However, the unconditional approach ignores the practitioner to practitioner variability (the variation in the $CARL_{IN}$ distribution). To this end, Saleh et al[2] used the *EPC* and bootstrapping to design the EWMA chart when parameters are estimated. The *EPC* does not ignore the variation in the $CARL_{IN}$ distribution but controls it with a high probability in the form of a prediction interval. The *EPC* was popularized by Jones and Steiner[8] and Gandy and Kvaloy[6]; since then, the *EPC* and the associated bootstrap approach have been used by many authors. We mention, among others, Saleh et al,[9] Aly et al,[10] Faraz et al,[11,12] and Hu and Castagliola.[13] However, bootstrapping is computer intensive and may be somewhat difficult to apply in practice. This is also exacerbated by the fact that, even though the underlying problem and the chart performance specifications may be the same, repeated applications of the bootstrap approach would almost surely result in different adjusted limits and can lead to comparability issues. Hence, it is not surprising that, with the exception of Faraz et al[11] and Hu and Castagliola,[13] the authors who have used the bootstrap approach did not provide or show tables of their new charting constants. Each of the Hu and Castagliola[13] charting constants was found by running the bootstrap approach 100 times and averaging the results. On an average computer, this takes a lot of time. Hence, without these tables, coming up with the charting constant can be frustrating for a practitioner.

Under the assumption that the process output is normally distributed, bootstrapping is not necessary to apply the *EPC*. For example, Goedhart et al[14-16] provided analytical results in the form of numerical solutions and approximations, for the Shewhart charts for the mean, and provided tables for the charting constants. But, for the EWMA chart, such analytical approximations are difficult to obtain because the charting statistics are dependent. Consequently, this paper presents a different method of adjusting the Phase II control limits according to the *EPC*, which guarantees, with a specified high confidence, that the $CARL_{IN}$ of the EWMA chart exceeds a nominal ARL_0. Our approach is based on the simple idea of approximating the $CARL_{IN}$ distribution by an empirical distribution, which is obtained by generating many Phase I subgroups, and using the Markov Chain to calculate the corresponding $CARL_{IN}$ values. It will be seen that this approach requires less computational effort than the bootstrap approach, yet it produces results that are as accurate as some known analytical results. Thus, tables and graphs of the required charting constants are provided to help practitioners implement the EWMA chart with estimated parameters easily in practice. A program to implement our method, written in R, is available from the authors on request.

The paper is organized as follows. Section 2 introduces some notation and terminology used, gives an overview of the EWMA chart and the Markov Chain technique, and presents the estimators that are used to estimate the unknown process parameters. Section 3 evaluates the traditional EWMA chart in terms of the *EPC* and provides rough guidelines on the number of Phase I subgroups required to achieve a certain high proportion of high $CARL_{IN}$ values relative to a reasonable nominal value. Section 4 presents the new charting constants (adjusted control limits) so that the Phase II EWMA chart has a guaranteed nominal *IC* performance according to the *EPC*. Section 5 gives a detailed evaluation of the *IC* and *OOC* performance of the new constants (the *EPC* adjusted limits based Phase II EWMA chart) and compares it with the performance of the traditional Phase II EWMA chart with unadjusted limits (limits calculated for Case K) according to the *EPC*. Finally, a summary and some conclusions are given.

2 | EWMA CHART WITH ESTIMATED PARAMETERS

Let X_{ij}, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ denote the *IC* Phase I data from a normal distribution with an unknown mean μ_0 and an unknown standard deviation σ_0. For a smoothing constant $0 < \lambda \leq 1$, starting at sampling stage $i = m+1, m+2, \ldots$, the standardized plotting statistic for the Phase II EWMA chart with the estimated parameters is given by

$$Y_i = \lambda W_i + (1-\lambda)Y_{i-1} \tag{1}$$

where $W_i = \dfrac{\overline{X}_i - \widehat{\mu}_0}{\widehat{\sigma}_0/\sqrt{n}}$, $\overline{X}_i$ is the i^{th} Phase II sample mean, and $\widehat{\mu}_0$ and $\widehat{\sigma}_0$ are the Phase I estimators of the unknown parameters μ_0 and σ_0, respectively. It is also assumed that the Phase II data are normally distributed, and for generality, let μ and σ denote the mean and the standard deviation, respectively, of this distribution. In this paper, we use the estimators $\widehat{\mu}_0 = \dfrac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$, the grand mean (see Schoonhoven et al[17]), and $\widehat{\sigma}_0 = \sqrt{\dfrac{1}{m}\sum_{i=1}^{m} S_i^2} = S_p$, the pooled standard deviation estimator, where S_i^2 denotes the variance of the i^{th} Phase I sample. Among the commonly used estimators for σ_0, the pooled standard deviation estimator provides the lowest values of the mean squared error (Mahmoud et al[18]). In addition, as noted in Diko et al,[19] the corresponding unbiased version $\widehat{\sigma}_0 = S_p/c_4(m(n-1)+1)$(see Montgomery,[20] Schoonhoven et al[21,22]) is equivalent, because, $m(n-1)$is typically quite large in our applications and hence the constant $c_4(m(n-1)+1)$ is indistinguishable from 1.

We write the statistic W_i in its canonical form

$$W_i = \frac{1}{Q}\left(\gamma T_i + \delta - \frac{Z}{\sqrt{m}}\right) \tag{2}$$

where $T_i = \dfrac{\overline{X}_i - \mu}{\sigma/\sqrt{n}}$, $Q = \dfrac{S_p}{\sigma_0}$, $Z = \dfrac{\widehat{\mu}_0 - \mu_0}{\sigma_0/\sqrt{mn}}$, $\gamma = \dfrac{\sigma}{\sigma_0}$, and $\delta = \dfrac{\mu - \mu_0}{\sigma_0/\sqrt{n}}$. Note that the random variables T_i and Z are independent standard normal variables that are mutually independent and are also independent of Q. Because $m(n-1)S_p^2/\sigma_0^2 \sim \chi^2_{m(n-1)}$, Q is distributed as $\sqrt{\dfrac{\chi^2_{m(n-1)}}{m(n-1)}}$. For simplicity, we use the asymptotic (steady state) control limits

$$h = h(n, \lambda, L) = \pm L\sqrt{\frac{\lambda}{(2-\lambda)}}. \tag{3}$$

where L is the charting constant to be found for a given λ value and some chart design/performance metric. The performance metric is usually some property of the *IC* run-length distribution, eg, ARL_{IN}. For example, for a given value of λ and a nominal $ARL_{IN} = ARL_0$, when parameters are known, the L values can be found in Crowder[23] or in the R package "spc." Often, these L values for Case K are used to construct the Phase II EWMA when estimated parameters are used in the control limits. It is recognized in the literature that this is a problem in the sense of getting many more false alarms than nominally expected, particularly when the amount of Phase I data is small to moderately large. We provide some solutions for correcting this problem.

Performance of a control chart is often evaluated by the run length distribution and its associated characteristics, eg, the mean (the expected value), the standard deviation, and percentiles. The conditional run-length distribution is the run-length distribution that is calculated for given values of Q and Z for a given set of data obtained from a Phase I analysis. The expected value of this distribution, denoted *CARL*, is also a random variable with its own distribution. The expected value of the distribution of *CARL* is the unconditional *ARL*, denoted *ARL*. The conditional run-length distribution and the *CARL* of an EWMA chart may be calculated (approximated) using the Markov Chain method, see Brook and Evans[24] and Lucas and Saccucci,[25] among others. Thus, applying the Markov Chain method, conditionally on Q and Z, the *CARL* of the Phase II EWMA chart can be conveniently written as

$$\begin{aligned} &CARL \\ &= \underset{\sim}{v}'(I-P)^{-1}\underset{\sim}{u} \\ &= CARL(\delta, Q, Z, m, n, \lambda, L, t), \end{aligned} \tag{4}$$

where t (which is generally taken as an odd integer) represents the number of transient states in the state space of a Markov Chain, $\underset{\sim}{v}'$ is the $1 \times t$ row vector with one in the middle position (for an odd integer t, the middle position is unique) and *0* elsewhere, $\underset{\sim}{u}$ is a $t \times 1$ column vector of ones, I is the $t \times t$ identity matrix, $P = [p_{lk}]$ is the $t \times t$ "essential" (conditional) transition probability matrix and l, $k = -\frac{t-1}{2}, \ldots, 0, \ldots, \frac{t-1}{2}$.

The transition probabilities of the essential conditional transition probability matrix,p_{lk}, are calculated, under normality and conditional on Q and Z, as follows.

$$\begin{aligned} p_{lk} &= \Phi\left(Q\left(\frac{S_k + w/2 - (1-\lambda)S_l}{\lambda}\right) - \delta + \frac{Z}{\sqrt{m}}\right) - \Phi\left(Q\left(\frac{S_k - w/2 - (1-\lambda)S_l}{\lambda}\right) - \delta + \frac{Z}{\sqrt{m}}\right) \\ &= p_{lk}(\delta, Q, Z, m.n, \lambda, L, t) \end{aligned} \tag{5}$$

where Φ denotes the cumulative distribution function of a standard normal variable, $w = \frac{2h}{t} = w(n, \lambda, L, t)$, $S_f = -\frac{w}{2} + \left(2\left(\frac{t-1}{2} + f\right) + 1\right)\frac{w}{2} = S_f(n, \lambda, L, t)$ and $f = l, k$. More information on the derivation of result (5) can be found in Saleh et al.[9]

From Equation 4, for fixed m, n, δ, λ, and L, it is clearly seen that the *CARL* depends on the random variables Q and Z, and hence the *CARL* is a random variable. Saleh et al[2] studied the effect of m and Phase I estimates on the distribution of $CARL_{IN}$ (the *CARL* when $\sigma = 0$). They found that unless the Phase I parameter estimates are "close" to the true but unknown parameter values, the $CARL_{IN}$ values can vary widely and from the nominal ARL_0. However, a practitioner will almost never know where his/her estimates are in relation to the unknown process parameters. Thus, when parameters are estimated, using the charting constants for Case K to design Phase II EWMA charts is a risky proposition, because it can result in very low $CARL_{IN}$ values which will almost surely call into question the process monitoring regime. This risk can be somewhat reduced by increasing m. However, as will be seen in the next section, the value of m that is required to reduce the probability of low $CARL_{IN}$ values can be very large. Hence, many control charts in the recent literature with estimated parameters are now designed such that

$$P(CARL_{IN} > ARL_0) = 1 - p. \tag{6}$$

It follows that the ARL_0 is the $100p^{\text{th}}$ percentile value of the distribution of $CARL_{IN}$. This is the *EPC* approach that we use to evaluate and design the EWMA chart in the following sections.

3 | PERFORMANCE ASSESSMENT OF A STANDARD PHASE II EWMA CHART USING THE *EPC*

Recall that a standard Phase II EWMA chart uses the charting constants obtained in the known parameter case when parameter estimates are plugged in to form the Phase II EWMA chart. Jones et al[1] and Saleh et al[2] studied the performance of the standard Phase II EWMA chart. However, their performance evaluations and sample size recommendations were based on the ARL_{IN} and $SDCARL_{IN}$, respectively. Moreover, even though the $SDCARL_{IN}$ accounts for some of the practitioner to practitioner variability, it does so in a different way compared with the *EPC*. Unlike the $SDCARL_{IN}$, the *EPC* approach does not only take into account the variability of the $CARL_{IN}$ distribution, it also considers the shape and the skewness. Hence, in this paper, we use the *EPC* approach to study the same traditional Phase II EWMA charts that were considered by Jones et al[1] and Saleh et al.[2] This allows us to compare and contrast our results with their results. Note also that Epprecht et al[4] used the *EPC* and its associated *CFAR* distribution to assess the performance of the Shewhart *S* and S^2 charts. Here, we use the more natural $CARL_{IN}$ distribution.

Consider again the *EPC* given in Equation 6, which can be re-written as

$$P(CARL_{IN}(Q, Z, m, n, \lambda, L, t) \leq ARL_0) = p.$$

Thus, for a given$p \in (0,1)$ and m, n, λ, L, t, we want to find the $100p^{\text{th}}$ percentile, $CARL_{IN,p}$, of the distribution of $CARL_{IN}(Q, Z, m, n, \lambda, L, t)$. Once found, $CARL_{IN,p}$ is compared with ARL_0, which is the theoretical value that must be exceeded, in an application, with a high probability $1-p$. The comparison between the $CARL_{IN,p}$ and ARL_0 will be based on the percentage difference (PD), which we define as $PD = \frac{CARL_{IN,p} - ARL_0}{ARL_0} \times 100$. The algorithm for the evaluation of the traditional Phase II EWMA using the EPC is given in Appendix B.

Table 1 shows the $CARL_{IN,p}$ values of the standard Phase II EWMA charts for $ARL_0 = 100,200,370,500$; $n = 5$ and different combinations of λ, m, and p. From Table 1, it can be seen that when m is small, the *PD* values (shown in the brackets in each cell) are very high in absolute values. This means, for example, for $\lambda = 0.1$, $m = 30$, $p = 0.05$ and $ARL_0 = 500$, we have $CARL_{IN,p} = 50$, which is *90%* below ($PD = -90\%$) the nominal $ARL_0 = 500$. Thus, in this case, we expect the $CARL_{IN}$ of the chart to be at least 50 with 95% probability (and conversely, the $CARL_{IN}$ of the chart to be at most 50 with 5% probability). Ideally, we would like the chart to deliver at least a large $CARL_{IN}$ value (say $ARL_0 = 500$) with 95% certainty. The value $CARL_{IN,p} = 50$ is too low, and the risk of getting a number that low is very high. It can also be seen that when m increases, the $CARL_{IN,p}$ values increase to within *6%* less than the nominal ARL_0 values. The convergence is faster for $\lambda = 0.5$ than for $\lambda = 0.1$. Furthermore, it can be seen that larger values of p or/and λ are associated with larger $CARL_{IN,\ p}$ values, improving the results slightly. Thus, when parameters are estimated, small $CARL_{IN}$ values (ie, the $CARL_{IN}$ values that are less than the ARL_0) occur more often than desired, while high $CARL_{IN}$ values (the $CARL_{IN}$ values above ARL_0). This is not acceptable. Table 1 also allows us to make rough recommendations about the number of Phase I subgroups m required to achieve adequate Phase II *EPC* performance. These recommendations are summarized in Table 2, and they are compared with the Jones et al[1] recommendation (that were based on the ARL_{IN} criteria) and Saleh et al[2] recommendations (that were based on $SDCARL_{IN}$ criteria) in Table 3.

Table 2 shows the number of subgroups m required to guarantee that the $CARL_{IN}$ exceeds $CARL_{IN,p}$ by a certain specified high probability $(1-p)$. Mathematically, this is written as

$$\begin{gathered} P\left(CARL_{IN} > CARL_{IN,p}\right) \geq 1-p \\ P(CARL_{IN} > ARL_0(1-\varepsilon)) \geq 1-p \end{gathered} \tag{7}$$

where $\varepsilon \geq 0\%$ is a nominally specified *PD* value. Note that $\varepsilon \geq 0$ because in general $CARL_{IN,p} < ARL_0$ (see Table 1). Note also that if $\varepsilon = 0\%$, then $CARL_{IN,p} = ARL_0$, and therefore Equation 7 reduces to Equation 6.

Looking at Table 2, for fixed ARL_0, λ, and p, it can be seen that decreasing ε from 20% to 0% increases the number of Phase I subgroups m required to achieve adequate *IC EPC* performance. It can also be seen that for fixed ARL_0, λ, and ε, decreasing p from 0.10 to 0.05 increases the value of m. Thus, decreasing ε or p or both improves the *IC* chart performance, while increasing ε or p or both degrades the *IC* chart performance. This also shows the flexibility of the *EPC* formulation (Equation 7), which can be used to improve the *IC* chart performance or to balance it with the *OOC* chart performance by manipulating ε or p or both. Later, we will provide an example of how this balance can be achieved. Table 3 compares our recommendations with the Jones et al[1] and Saleh et al[2] recommendations.

From Table 3, it can be seen that for $p = 0.05, 0.10$; $\varepsilon = 0\%$, $n = 5$, and all λ, it will take more than 10 000 Phase I subgroups to guarantee (with a high probability) that the nominal ARL_0 value will be exceeded. Thus, based on the *EPC*, it is seen that significantly more Phase I data are required than previously recommended by both Jones et al[1] and Saleh et al.[2] Furthermore, for the *EPC* approach, it can be seen that when $CARL_{IN,p}$ is $\varepsilon = 10\%$ or $\varepsilon = 20\%$ below the ARL_0; a large number of subgroups is still required to guarantee with high certainty that $CARL_{IN} > CARL_{IN,p}$. Moreover, small λ values require more data than larger λ values. This agrees with the findings of Jones et al,[1] but it is in contrast with the findings of Saleh et al.[2]

TABLE 1 The 5th and the 10th ($P = 0.05, 0.10$) percentiles of the $CARL_{IN}$ distribution as a function of m for $\lambda = 0.1, 0.5$, $n = 5$ and $ARL_0 = 100{,}200{,}370{,}500$

		$ARL_0 = 100$ ($L = 2.148$)		$ARL_0 = 200$ ($L = 2.454$)		$ARL_0 = 370$ ($L = 2.702$)		$ARL_0 = 500$ ($L = 2.815$)	
λ	m	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$
0.1	30	24 (−76%)	30 (−70%)	34 (−83%)	45 (−78%)	44 (−88%)	61 (−84%)	50 (−90%)	71 (−86%)
	50	32 (−68%)	39 (−61%)	48 (−76%)	63 (−69%)	69 (−81%)	92 (−75%)	84 (−83%)	112 (−78%)
	100	47 (−53%)	55 (−45%)	76 (−62%)	92 (−54%)	115 (−69%)	141 (−62%)	141 (−72%)	179 (−64%)
	400	72 (−28%)	77 (−23%)	135 (−33)	146 (−27%)	232 (−37%)	257 (−31%)	298 (−40%)	336 (−33%)
	500	75 (−25%)	79 (−21%)	143 (−29%)	152 (−24%)	247 (−33%)	269 (−27%)	324 (−35%)	354 (−29%)
	600	78 (−22%)	81 (−19%)	147 (−27%)	156 (−22%)	257 (−31%)	278 (−25%)	341 (−32%)	370 (−26%)
	900	81 (−19%)	83 (−17%)	157 (−22%)	164 (−18%)	283 (−24%)	298 (−20%)	376 (−25%)	396 (−21%)
	1000	81 (−19%)	84 (−16%)	160 (−20%)	166 (−17%)	288 (−22%)	299 (−19%)	381 (−24%)	404 (−19%)
	1500	84 (−16%)	86 (−14%)	168 (−16%)	172 (−14%)	303 (−18%)	314 (−15%)	406 (−19%)	422 (−16%)
	2000	86 (−14%)	87 (−13%)	171 (−15%)	175 (−13%)	313 (−15%)	322 (−13%)	421 (−16%)	434 (−13%)
	4000	88 (−12%)	89 (−11%)	178 (−11%)	181 (−10%)	329 (−11%)	335 (−9%)	444 (−11%)	452 (−10%)
	6000	89 (−11%)	90 (−10%)	181 (−10%)	183 (−9%)	336 (−9)	341 (−8%)	454 (−9%)	460 (−8%)
	10000	90 (−10%)	91 (−9%)	183 (−9%)	185 (−8%)	342 (−8)	345 (−7%)	462 (−8%)	467 (−7%)
		$ARL_0 = 100$ ($L = 2.534$)		$ARL_0 = 200$ ($L = 2.777$)		$ARL_0 = 370$ ($L = 2.978$)		$ARL_0 = 500$ ($L = 3.071$)	
λ	m	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$
0.5	30	34 (−66%)	41 (−59%)	58 (−71%)	72 (−64%)	87 (−77%)	111 (−70%)	111 (−78%)	143 (−71%)
	50	46 (−54%)	52 (−48%)	80 (−60%)	94 (−53%)	126 (−66%)	154 (−58%)	161 (−68%)	198 (−60%)
	100	59 (−41%)	65 (−35%)	106 (−47%)	120 (−40%)	182 (−51%)	206 (−44%)	239 (−52%)	272 (−46%)
	400	79 (−21%)	82 (−18%)	152 (−24%)	160 (−20%)	267 (−28%)	284 (−23%)	355 (−29%)	379 (−24%)
	500	81 (−19%)	84 (−16%)	155 (−23%)	163 (−19%)	278 (−25%)	295 (−20%)	372 (26%)	394 (−21%)
	600	82 (−18%)	85 (−15%)	159 (−21%)	167 (−17%)	290 (−22%)	303 (−18%)	385 (−23%)	403 (−19%)
	900	85 (−15%)	88 (−12%)	167 (−17%)	172 (−14%)	302 (−18%)	314 (−15%)	400 (−20%)	419 (−16%)
	1000	86 (−14%)	88 (−12%)	168 (−16%)	174 (−13%)	304 (−18%)	316 (−15%)	405 (−19%)	424 (−15%)
	1500	88 (−12%)	90 (−10%)	173 (−14%)	178 (−11%)	316 (−15%)	327 (−12%)	423 (−15%)	437 (−13%)
	2000	90 (−10%)	91 (−9%)	177 (−12%)	181 (−10%)	325 (−12%)	333 (−10%)	434 (−13%)	447 (−11%)
	4000	92 (−8%)	94 (−6%)	183 (−9%)	186 (−7%)	336 (−9%)	343 (−7%)	452 (−10%)	460 (−8%)
	6000	94 (−6%)	95 (−5%)	186 (−7%)	188 (−6%)	342 (−8%)	348 (−6%)	460 (−8%)	467 (−7%)
	10000	95 (−5%)	95 (−5%)	189 (−6%)	191 (−5%)	348 (−6%)	352 (−5%)	468 (−6%)	475 (−5%)

TABLE 2 Minimum m required for $CARL_{IN,p}$ to be $\varepsilon = 0$ %, 10 %, 20% below the nominally $ARL_0 = 100{,}200{,}370{,}500$ for $n = 5; \lambda = 0.1, 0.5$ and $p = 0.05, 0.10$

		$ARL_0 = 100$		$ARL_0 = 200$		$ARL_0 = 370$		$ARL_0 = 500$	
	ε	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$	$P = 0.05$	$P = 0.10$
$\lambda = 0.1$	0%	>10 000	>10 000	>10 000	>10 000	>10 000	>10 000	>10 000	>10 000
	10%	10 000	6000	6000	4000	6000	4000	6000	4000
	20%	900	600	1000	900	1500	900	1500	1000
$\lambda = 0.5$	0%	>10 000	>10 000	>10 000	>10 000	>10 000	>10 000	>10 000	>10 000
	10%	2000	1500	4000	2000	4000	2000	4000	2000
	20%	500	400	600	400	900	500	900	600

TABLE 3 Recommended minimum number of phase I subgroups when $n = 5$ and $ARL_0 = 200$

	λ			
	0.1	**0.2**	**0.5**	**1**
Jones et al[1] (marginal ARL criteria)	400	300	200	100
Saleh et al[2] $SDCARL_{IN}$ criteria	600	700	900	1000
This paper EPC criteria with $p = 0.10$ and $\varepsilon = 20\%$	900	600	400	370
This paper EPC criteria with $p = 0.05$ and $\varepsilon = 20\%$	1000	700	600	560
This paper EPC criteria with $p = 0.10$ and $\varepsilon = 10\%$	4000	3000	2000	1500
This paper EPC criteria with $p = 0.05$ and $\varepsilon = 10\%$	6000	5000	4000	2500
This paper EPC criteria with $p = 0.05, 0.10$ and $\varepsilon = 0\%$	>10 000	>10 000	>10 000	>10 000

4 | ADJUSTMENT OF THE STANDARD PHASE II EWMA CHART LIMITS FOR GUARANTEED CONDITIONAL PERFORMANCE

We have seen that, to achieve adequate *EPC* performance, a very high number of Phase I subgroups is required when using the standard Phase II EWMA chart limits. In practice, it may be difficult and expensive to come up with these high Phase I subgroup numbers. Thus, for a given amount of Phase I data (number of Phase I subgroups, with a fixed sample size), the control limits need to be adjusted.

Consider again the *EPC*: $P(CARL_{IN}(Q, Z, m, n, \lambda, L, t) > ARL_0(1 - \varepsilon)) \geq 1 - p$, which is equivalent to stating that the *cdf* of $CARL_{IN}(Q, Z, m, n, \lambda, L, t)$ at ARL_0 must be less than or equal to p. Then, given ε, p, ARL_0, m, n, λ, and t, we want to solve this equation for L. Because a closed-form analytical expression for the *cdf* of $CARL_{IN}$ is not available, a formula to

TABLE 4 L values that guarantee that $P(CARL_{IN} > ARL_0) = 0.90$ for the EWMA $\overline{X}$ chart for $n = 5$; $m = 30, 50, 100, 300, 1000$; $\lambda = 0.1, 0.2, 0.5, 1$; $\varepsilon = 0\%$ and $ARL_0 = 100, 200, 370, 500$

		$p = 0.10$			
ARL_0	m	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.5$	$\lambda = 1$
100	30	3.09	3.01	2.92	2.88
	50	2.79	2.79	2.79	2.79
	100	2.50	2.62	2.71	2.72
	300	2.32	2.48	2.62	2.66
	1000	2.23	2.42	2.58	2.62
	Known parameter	2.148	2.360	2.534	2.576
200	30	3.49	3.34	3.20	3.13
	50	3.16	3.12	3.08	3.03
	100	2.86	2.92	2.96	2.96
	300	2.63	2.77	2.87	2.89
	1000	2.53	2.70	2.83	2.85
	Known parameter	2.454	2.636	2.777	2.807
370	30	3.78	3.59	3.43	3.34
	50	3.46	3.38	3.30	3.24
	100	3.16	3.16	3.16	3.16
	300	2.89	2.99	3.09	3.09
	1000	2.78	2.92	3.04	3.05
	Known parameter	2.702	2.859	2.978	3.000
500	30	3.92	3.70	3.54	3.44
	50	3.59	3.49	3.40	3.34
	100	3.29	3.28	3.26	3.26
	300	3.02	3.10	3.18	3.18
	1000	2.90	3.03	3.13	3.14
	Known parameter	2.815	2.962	3.071	3.090

calculate the $CARL_{IN}$is. Our approach is to generate the empirical distribution of $CARL_{IN}$ using different values of L in the interval [L, ∞), starting from the Case K L value towards infinity. For each empirical distribution, the $CARL_{IN,p}$ value is calculated. The first value of L for which $CARL_{IN,p} > ARL_0(1-\varepsilon)$ is chosen to be the solution. A step-by-step algorithm for finding L is given in Appendix C. Like the other algorithms we presented, this algorithm requires an approximation of the $CARL_{IN}$ distribution, via the empirical distribution. In our view, this is what gives it an edge over the bootstrap algorithm used in Saleh et al[2] and others, which requires more computational effort.

Table 4 gives the L values that guarantee, with $(1-p)$% probability, that the $CARL_{IN}$ will exceed a specified lower bound ARL_0. Looking at Table 4 for $ARL_0 = 370$, $\lambda = 1$ and $m = 50, 100, 300, 1000$, it can be seen that our constants $L = 3.24, 3.16, 3.09, 3.05$ are exactly equal to those in Goedhart et al.[14,16] The constants in Goedhart et al[14,16] were obtained analytically and are regarded as an improvement to the computationally intensive bootstrap approach. This

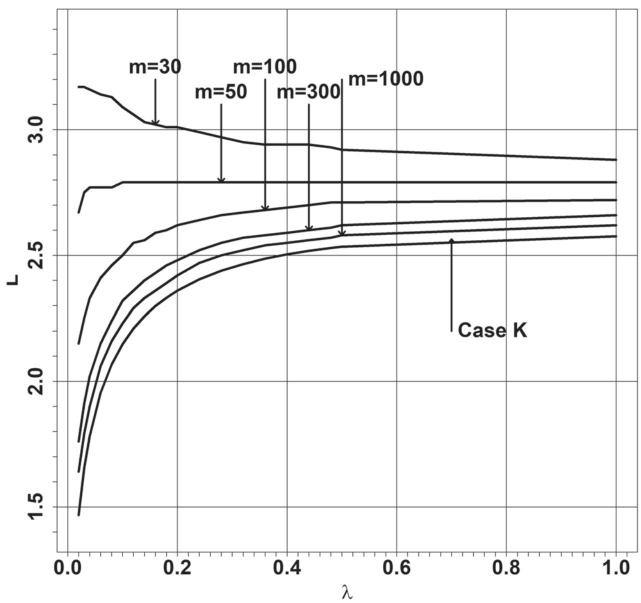

FIGURE 1 Graphs of the unadjusted (case K) and adjusted L values for $0.02 \le \lambda \le 1$; $ARL_0 = 100$ and $n = 5$. The adjusted L values were generated to guarantee $P(CARL_{IN} > ARL_0) = 0.90$ for $m = 30,50,100,300,1000$

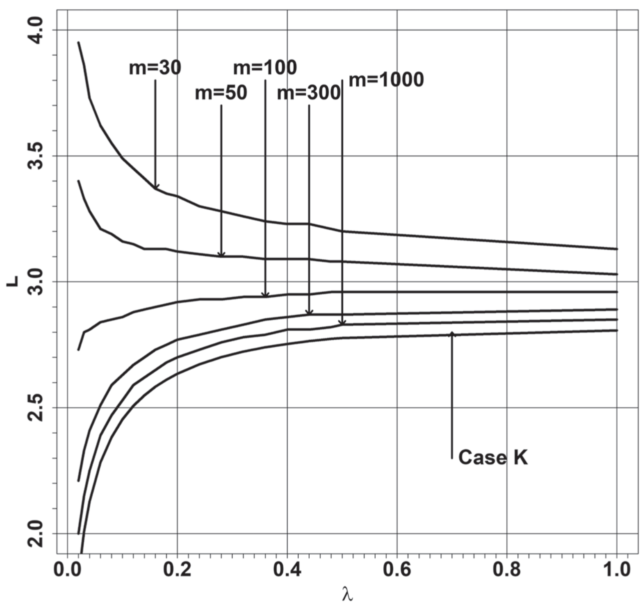

FIGURE 2 Graphs of the unadjusted (case K) and adjusted L values for $0.02 \le \lambda \le 1$; $ARL_0 = 200$ and $n = 5$. The adjusted L values were generated to guarantee $P(CARL_{IN} > ARL_0) = 0.90$ for $m = 30,50,100,300,1000$

validates our method. In addition to Table 4 for $ARL_0 = 100{,}200{,}370{,}500$, we have generated four figures in which the practitioner may find his/her constant L given its own m and λ by means of interpolation. These are shown below.

Looking at Figures 1–4, for any given ARL_0, m and λ values, it can be seen that the adjusted L values are all greater than the corresponding Case K L values. It can also be seen that, for a given ARL_0 and λ, the adjusted L values decrease as m increases and converge to the known parameter (unadjusted/standard) L value. Consequently, Phase II EWMA chart that are designed using the new L values will have wider control limits, and this will lead to an improved *IC* performance than the charts whose design uses the Case K L values. This improved *IC* performance, that is widening the limits, can lead to some deterioration of the *OOC* chart performance. This has been noted in the literature (see, eg, Goedhart et al[15]) as the price to pay for satisfactory nominal *IC* chart performance with a high probability. However, it is possible with our approach to relax the *IC* behavior of the EWMA chart. This can be done by increasing ε or p or

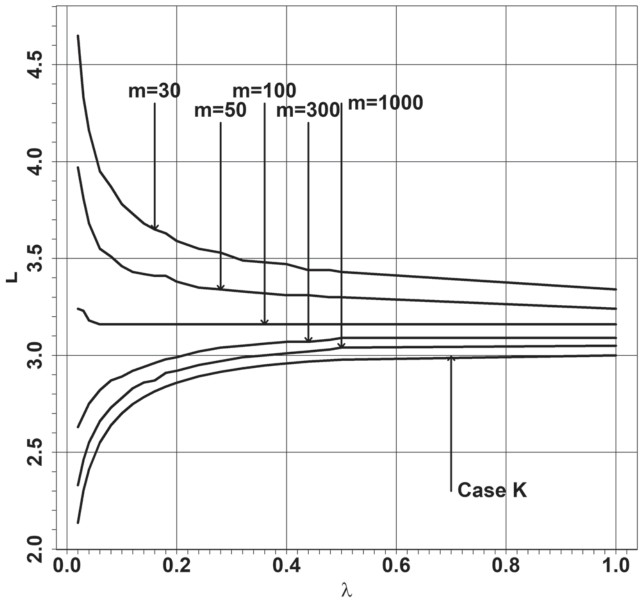

FIGURE 3 Graphs of the unadjusted (case K) and adjusted L values for $0.02 \leq \lambda \leq 1$; $ARL_0 = 370$ and $n = 5$. The adjusted L values were generated to guarantee $P\ (CARL_{IN} > ARL_0) = 0.90$ for $m = 30{,}50{,}100{,}300{,}1000$

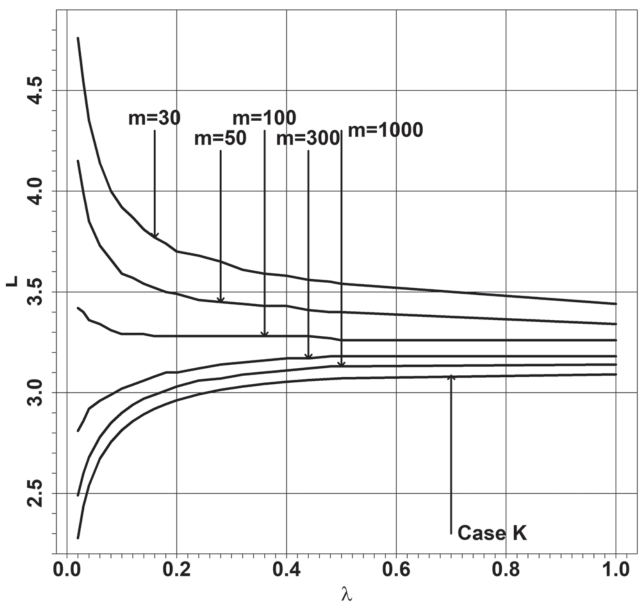

FIGURE 4 Graphs of the unadjusted (case K) and adjusted L values for $0.02 \leq \lambda \leq 1$; $ARL_0 = 500$ and $n = 5$. The adjusted L values were generated to guarantee $P\ (CARL_{IN} > ARL_0) = 0.90$ for $m = 30{,}50{,}100{,}300{,}1000$

TABLE 5 The $CARL_{IN,p}$ values for the *IC* and *OOC* performance of the EWMA $\overline{X}$chart with the adjusted ($ARL_0 = 200, p = 10\%$)and unadjusted ($ARL_0 = 200$) limits for $n = 5$; $m = 50,100$; $\lambda = 0$, 0.1, 0.2, 0.5, 1; $\delta = 0$, 0.25, 0.5, 1, 1.5 and some percentiles

m = 50								
Perc	**$\lambda = 0.1$**		**$\lambda = 0.2$**		**$\lambda = 0.5$**		**$\lambda = 1$**	
	Unadjusted	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**
$\delta = 0$								
0.05	48	139	60	147	79	166	89	169
0.10	**62**	**202**	**75**	**198**	**92**	**205**	**104**	**205**
0.25	92	374	105	322	121	283	134	269
0.50	133	696	146	518	166	411	178	369
0.75	177	1138	197	786	221	581	246	520
0.90	220	1694	252	1140	295	801	327	726
0.95	251	2142	293	1453	346	981	392	889
$\delta = 0.25$								
0.05	27	50	29	57	43	82	70	122
0.10	31	63	35	73	54	103	82	144
0.25	43	102	50	115	77	161	108	199
0.50	**66**	**198**	**81**	**212**	**112**	**254**	**149**	**285**
0.75	108	452	127	401	164	399	207	415
0.90	163	916	184	704	226	596	278	586
0.95	196	1295	222	928	267	776	333	717
$\delta = 0.5$								
0.05	16	23	15	24	21	35	42	69
0.10	17	27	17	27	24	42	48	82
0.25	21	34	22	37	33	58	64	115
0.50	**26**	**48**	**29**	**55**	**47**	**90**	**89**	**167**
0.75	36	76	41	89	69	143	126	245
0.90	51	125	61	148	102	228	174	341
0.95	65	182	79	214	125	302	209	425
$\delta = 1$								
0.05	8	11	7	9	8	10	15	22
0.10	9	12	8	10	8	12	17	26
0.25	10	13	9	11	10	14	21	34
0.50	**11**	**15**	**10**	**14**	**12**	**19**	**28**	**46**
0.75	12	18	12	17	16	25	38	66
0.90	14	21	14	20	20	33	51	90
0.95	15	23	15	23	23	40	62	109
$\delta = 1.5$								
0.05	6	7	5	6	4	5	6	9
0.10	6	8	5	6	4	6	7	10
0.25	6	8	5	7	5	6	8	12
0.50	**7**	**9**	**6**	**7**	**6**	**7**	**10**	**16**
0.75	7	10	6	8	7	9	13	21

(Continues)

TABLE 5 (Continued)

m = 50								
Perc	**$\lambda = 0.1$**		**$\lambda = 0.2$**		**$\lambda = 0.5$**		**$\lambda = 1$**	
	Unadjusted	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**
0.90	8	11	7	9	8	10	17	28
0.95	8	11	7	10	8	12	19	33
m = 100								
Perc	**$\lambda = 0.1$**		**$\lambda = 0.2$**		**$\lambda = 0.5$**		**$\lambda = 1$**	
	Unadjusted	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**
$\delta = 0$								
0.05	75	152	90	173	107	171	117	177
0.10	**92**	**200**	**106**	**208**	**121**	**196**	**131**	**199**
0.25	123	290	131	267	145	242	157	243
0.50	155	399	163	344	178	307	192	306
0.75	187	523	202	443	219	390	237	385
0.90	221	650	239	547	268	486	287	472
0.95	240	734	264	619	303	548	326	547
$\delta = 0.25$								
0.05	33	50	38	57	58	86	86	133
0.10	38	58	43	68	66	100	98	152
0.25	49	79	56	94	86	133	122	189
0.50	**67**	**119**	**79**	**138**	**113**	**182**	**151**	**242**
0.75	95	188	111	213	148	248	193	315
0.90	134	302	149	304	189	328	239	398
0.95	162	387	174	374	217	388	267	453
$\delta = 0.5$								
0.05	18	23	18	24	26	36	51	74
0.10	19	25	20	26	30	41	57	85
0.25	22	30	24	32	37	53	71	106
0.50	**26**	**37**	**30**	**42**	**47**	**70**	**90**	**137**
0.75	32	48	38	56	61	96	115	178
0.90	40	63	49	74	78	127	144	225
0.95	46	76	58	90	93	151	165	262
$\delta = 1$								
0.05	9	11	8	9	7	11	17	23
0.10	9	11	8	10	9	11	19	26
0.25	10	12	9	11	11	13	23	32
0.50	**11**	**13**	**10**	**12**	**12**	**16**	**28**	**40**
0.75	12	15	11	13	15	19	35	50
0.90	13	16	12	15	17	23	43	62
0.95	14	17	13	176	19	25	47	70
$\delta = 1.5$								
0.05	6	7	5	6	5	5	7	9

(Continues)

TABLE 5 (Continued)

m = 100								
Perc	**$\lambda = 0.1$**		**$\lambda = 0.2$**		**$\lambda = 0.5$**		**$\lambda = 1$**	
	Unadjusted	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**	**Unadjusted**	**Adjusted**
0.10	6	7	5	6	5	5	8	10
0.25	6	8	5	6	5	6	9	12
0.50	**7**	**8**	**6**	**7**	**6**	**7**	**10**	**14**
0.75	7	9	6	7	6	7	12	17
0.90	8	9	7	8	7	8	14	20
0.95	8	9	7	8	7	9	16	22

both in Equation 7. As will be seen, in the next section, the result will be less wider adjusted limits, which will improve the *OOC* performance.

5 | *IC* AND *OOC* PERFORMANCE ANALYSIS AND A COMPARISON OF THE ADJUSTED AND THE UNADJUSTED LIMITS

Using the bootstrap approach, Saleh et al[2] came up with an EWMA chart such that $P(CARL_{IN} > 200) = 0.90$ for $\lambda = 0.1$, $m = 50$and $n = 5$. Following this, they evaluated the *IC* and *OOC* conditional performance of this chart. However, their performance evaluations were done only for this chart and were limited only to $\delta = 0$ and $\delta = 1$. In this section, we make a much more detailed evaluation and comparison between the performance of the EWMA charts with the proposed adjusted limits and that of the standard (unadjusted) limits chart, for various shifts δ. We also compare our results with the performance results of the bootstrap based (adjusted limits) EWMA chart in Saleh et al.[2] Furthermore, we use the flexibility of the *EPC* formulation in Equation 7 to adjust the trade-off between the *IC* and *OOC* performance of the EWMA chart. By trade-off, we mean, for example, sacrificing a little *IC* performance for a better *OOC* performance.

Table 5 shows the $CARL_{IN,p}$ values for various combinations of p, λ, δ, and m for both adjusted and unadjusted limits. Again, for given λ and $\delta = 0$, the adjusted limits were obtained such that $P(CARL_{IN} > 200) = 1 - p = 0.90$ (so that the 10th percentiles of the $CARL_{IN}$ distribution should be close to 200) while the unadjusted limits were obtained from the R package "spc," such that $ARL_0 = 200$, in the known parameters case. Looking at Table 5, for $\delta = 0$and all λ, it can be seen that the *IC* performance of the chart with the adjusted limits is as specified. For example, for $m = 50$, $p = 10\%$ and $\lambda = 0.1, 0.2, 0.5, 1$, it can be seen (see the bolded row at perc = 10%) that $CARL_{IN,p} = 202{,}198{,}205{,}205$; respectively, and for $m = 100$, we have $CARL_{IN,p} = 200{,}208{,}196{,}199$. All of these $CARL_{IN,p}$ are very close to the nominal $ARL_0 = 200$. Besides, for $\delta = 0$and all λ, p, m values, the values for the adjusted limits are always higher than the corresponding unadjusted limit. So, for all percentiles (*perc*), the adjusted limits charts always guarantee, with high probability (close to the nominal), larger $CARL_{IN}$ values compared with the unadjusted limits charts. Thus, the good *IC* performance of the adjusted limits charts is not only limited to $p = 10\%$, but extends over the entire range of *perc*'s.

However, as mentioned before, because the adjusted limits are wider, they can be insensitive to true process shifts compared with the unadjusted limits. We explore this for the cases when $\delta = 0.25, 0.5, 1, 1.5$. Looking at Table 5 for $m = 50$, small shifts $\delta = 0.25$, 0.5and all λ values, it can be seen that the medians (see the bolded rows at perc = 10%) of the *CARL* distributions for the unadjusted and the adjusted limits charts are radically different. The largest difference occurs at $\lambda = 0.1$, while the smallest difference occurs at $\lambda = 1$. Decreasing *perc* and/or increasing m reduces the differences slightly, but the pattern remains the same. Thus, for a small shift $\delta \leq 0.50$, the *OOC* chart performance of the adjusted limits EWMA chart is not as good as that of the unadjusted limits charts, particularly when $\lambda = 0.1$. But of course the point is that the *IC* performance of the unadjusted limits based chart is a much bigger problem. However, for larger shifts, such as $\delta = 1$ or $\delta = 2$ and for all *perc* and λ values, the $CARL_{IN,p}$ values for the unadjusted and the adjusted limit charts are quite close. This is even more so when $m = 100$. Thus, for moderate to large values of δ, the *OOC* chart performance of the adjusted limits EWMA chart is comparable to that of the EWMA chart with the unadjusted limits or the limits for the known parameter case.

Note that, in the literature (eg, Saleh et al[2]), authors who use the bootstrap approach often compare the *OOC* behavior of the unadjusted and the adjusted limits charts solely on the basis of a shift of size $\delta = 1$. Figure 5 shows the

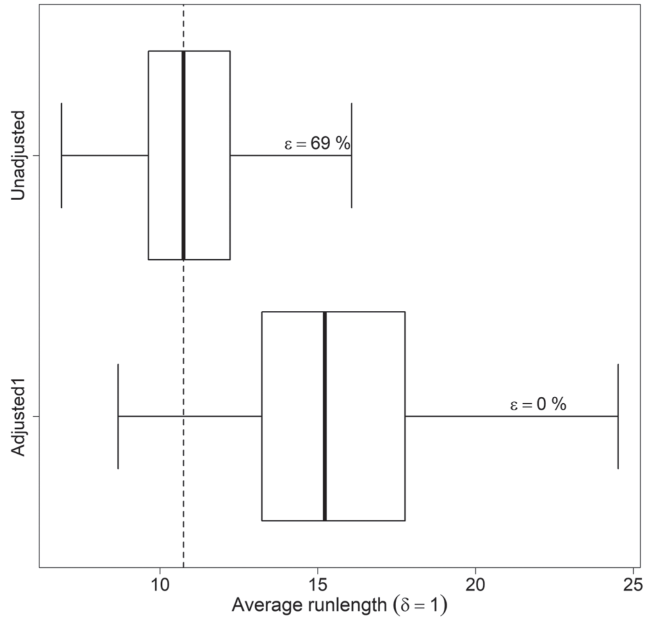

FIGURE 5 Boxplots of the *CARL* distribution when $\delta = 1, \lambda = 0.1, p = 10\%, ARL_0 = 200, m = 50$ and $n = 5$

boxplots for the *OOC CARL* distributions of the EWMA chart with the adjusted and the unadjusted limits for $\lambda = 0.1$, $\delta = 1$, $m = 50$ and $n = 5$. Based on Figure 5, it has always been concluded that the *OOC* performance of the bootstrap adjusted limits is not radically different from that of the unadjusted limits. However, we have shown through the $CARL_{IN,p}$ values, in Table 5, that this only occurs when δ is moderate to large. Therefore, widening the control limits by the *EPC* criterion makes them a little insensitive to small process shifts but guarantees a nominal performance with high probability. This may be the trade-off one has to accept. However, it is possible to adjust this trade-off to get a better *OOC* performance. This can be done by sacrificing a bit of *IC* performance. For fixed *p*, the *IC* performance can be sacrificed by increasing ε in Equation 7, while for fixed ε, it can be reduced by increasing *p* in Equation 7. To illustrate the former, Figures 6 and 7 show the boxplots for the *IC* and *OOC CARL* distributions of the EWMA chart, respectively, for $\lambda = 0.1$; $ARL_0 = 200$; $\varepsilon = 0\ \%\ ,\ 35\ \%\ ,\ 69\ \%$; $p = 0.10$; $m = 50$ and $n = 5$.

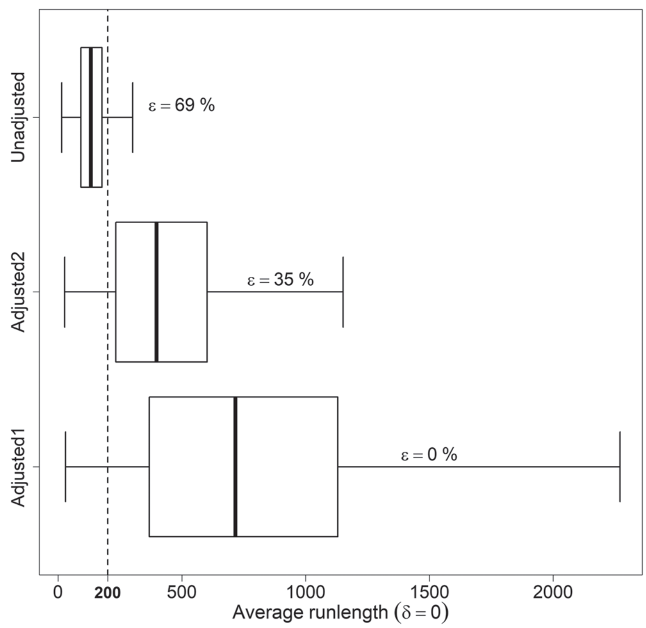

FIGURE 6 Boxplots of the *CARL* distribution when $\delta = 0, \lambda = 0.1, p = 10\%, ARL_0 = 200, m = 50$ and $n = 5$

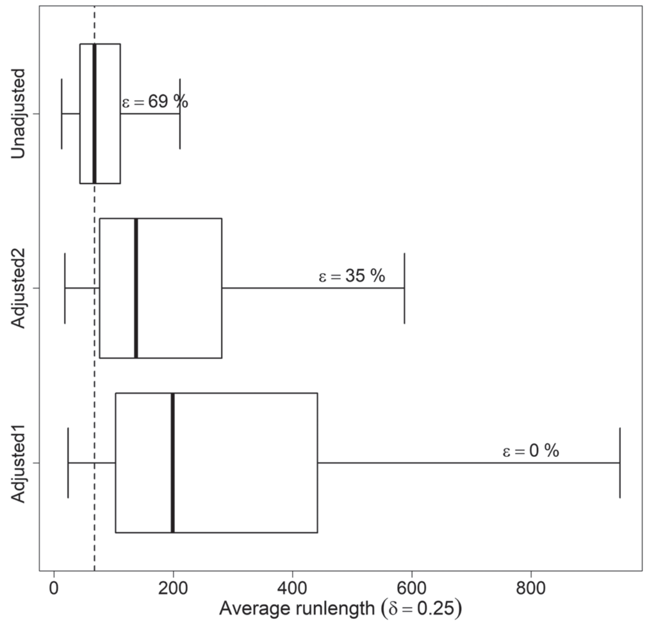

FIGURE 7 Boxplots of the *CARL* distribution when $\delta = 0.25, \lambda = 0.1, p = 10\%, ARL_0 = 200, m = 50$ and $n = 5$

From Figure 6, it can be seen that increasing ε from *0%* to *35%* leads to a slight loss of the *IC* performance. For example, when $\varepsilon = 35\%$, the proportion of $CARL_{IN}$ values that are less than *200* is *20%*. But, this is still way better than the *85%* that occurs when the unadjusted limits ($\varepsilon = 69\%$) are used. From Figure 7, it can also be seen that increasing ε from *0%* to *35%* leads to an improved *OOC* performance in the sense that the median for the *OOC* $CARL_{IN}$ distribution of $\varepsilon = 35\%$ is closer to the median (the dotted vertical line) for the *OOC* $CARL_{IN}$ distribution of $\varepsilon = 69\%$. Thus, by sacrificing a bit of the *IC* performance, it is possible to improve the EWMA charts ability to detect small shifts.

6 | SUMMARY AND CONCLUSIONS

We study the impact of practitioner to practitioner variability on the performance of the Phase II EWMA chart. As in Epprecht et al,[4] we use the *EPC* criterion to evaluate the performance of a Phase II EWMA chart with limits for the known parameter case and give recommendations about the required number of Phase I subgroups to achieve nominal performance. Our results show that in order to attain or exceed a specified lower bound of $CARL_{IN}$(given by ARL_0) with a specified high probability, more Phase I data are required than previously recommended by Saleh et al[2] and Jones et al[1] Moreover, consistently with Jones et al[1] but contrary to Saleh et al,[2] our results also show that smaller values of λ may require a larger number of Phase I subgroups, that is, more Phase I data.

Because it is expensive and sometimes impractical to get such large amount of Phase I data to estimate the process parameters and construct Phase II charts that guarantee a high probability of high $CARL_{IN}$'s under the *EPC*, the control limits are adjusted as a function of the available Phase I data. In this regard, where analytical methods could not be conveniently used, eg, for the *EWMA* chart or where normality cannot be assumed, the bootstrap approach has been an attractive choice. However, many SPC practitioners and researchers have felt that the bootstrap approach may be somewhat complex and have looked for an alternative. In this paper, we presented an alternative method that can be used instead of the bootstrap approach. Our method produces the same results as the bootstrap approach, but it is faster. Based on the new method, tables and the graphs of the adjusted charting constants are provided to help practitioners implement the Phase II *EWMA* chart with estimated parameters more easily in practice. The new charting constants are larger than the traditional ones commonly used for Case K. Thus, the *EWMA* charts constructed using these new constants have wider limits, particularly for small λ and/or m.

Adjusting the limits of the *EWMA* chart, using our new constants, guarantees with high probability that the $CARL_{IN}$ performance will be as nominally specified. However, there is some concern about the deterioration in the *OOC CARL* performance relative to using the unadjusted limits which are wider. This is of course true for all types of control charts with estimated parameters and has been observed, for example, for the Shewhart charts (see Goedhart et al[15]). The extent of the deterioration depends on the size of the shift $\delta$and m. For moderate to large shifts (say $\delta = 1$ and more), the difference in the *OOC CARL* performance between the adjusted and unadjusted limits is negligible. However, for small shifts (say $\delta = 0.25$, 0.50) and small m, the difference is not negligible. Thus, adjusting the control limits can make the chart somewhat insensitive to detecting small shifts. The insensitivity to small shifts may be improved by sacrificing some *IC* chart performance as illustrated in Figures 6 and 7. Nonetheless, it is important to keep in mind that the *IC* chart performance is perhaps the most important to have higher confidence in, so sacrificing some *OOC* performance may be the price one has to pay when a given amount of Phase I data are used to estimate the parameters to construct a control chart.

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REFERENCES

1. Jones LA, Champ CW, Rigdon SE. The performance of exponentially weighted moving average charts with estimated parameters. *Technometrics*. 2001;43(2):156-167. https://doi.org/10.1198/004017001750386279
2. Saleh NA, Mahmoud MA, Jones-Farmer LA, Zwetsloot I, Woodall WH. Another look at the EWMA control chart with estimated parameters. *J Qual Technol*. 2015;47(4):363-382. https://doi.org/10.1080/00224065.2015.11918140
3. Chakraborti S. Parameter estimation and design considerations in prospective applications of the X¯ chart. *J Appl Stat*. 2006;33(4):439-459.
4. Epprecht EK, Loureiro LD, Chakraborti S. Effect of the amount of phase I data on the phase II performance of S^2 and S control charts. *J Qual Technol*. 2015;47(2):139-155.
5. Albers W, Kallenberg WCM, Nurdiati S. Exceedance probabilities for parametric control charts. *Stat*. 2005;39(5):429-443.
6. Gandy A, Kvaloy JT. Guaranteed conditional performance of control charts via bootstrap methods. *Scand J Stat*. 2013;40(4):647-668.
7. Jones LA. The statistical design of EWMA control charts with estimated parameters. *J Qual Technol*. 2002;34(3):277-288.
8. Jones MA, Steiner SH. Assessing the effect of estimation error on the risk-adjusted CUSUM chart performance. *Int J Qual Health Care*. 2012;24(2):176-181.
9. Saleh NA, Mahmoud AM, Abdel-Salem A-SG. The performance of the adaptive exponentially weighted moving average control chart with estimated parameters. *Qual Reliab Eng Int*. 2013;29(4):95-606.
10. Aly AA, Saleh NA, Mahmoud MA, Woodall WH. A reevaluation of the adaptive exponentially weighted moving average control chart when parameters are estimated. *Qual Reliab Eng Int*. 2015;31(8):1611-1622.
11. Faraz A, Heuchenne C, Saniga E. The np chart with guaranteed in-control average run-lengths. *Qual Reliab Eng Int*. 2016;33(5):1057-1066.
12. Faraz A, Woodall WH, Heuchenne C. Guaranteed conditional performance of the S2 control chart with estimated parameters. *Int J Prod Res*. 2015;53(14):4405-4413.
13. Hu X, Castagliola P. Guaranteed conditional Design of the Median Chart with estimated parameters. *Quality and Reliability Engineering International*. 2017;33(8):1873-1884.
14. Goedhart R, Schoonhoven M, Does RJMM. Guaranteed in-control performance for the Shewhart X and Xbar control charts. *J Qual Technol*. 2017;49(2):155-171.
15. Goedhart R, da Silva MM, Schoonhoven M, et al. Shewhart control charts for dispersion adjusted for parameter estimation. *IISE Trans*. 2017;49(8):838-848.
16. Goedhart R, Schoonhoven M, Does RJMM. On guaranteed in-control performance for the Shewhart X and X¯ control charts. *J Qual Technol*. 2018;50(1):130-132.
17. Schoonhoven M, Nazir HZ, Riaz M, Does RJMM. Robust location estimators for the X¯ control chart. *J Qual Technol*. 2011;43(4):363-379.
18. Mahmoud AM, Henderson GR, Epprecht EK, Woodall WH. Estimating the standard deviation in quality-control applications. *J Qual Technol*. 2010;42(4):348-357.
19. Diko MD, Goedhart R, Chakraborti S, Does RJMM, Epprecht EK. Phase II control charts for monitoring process dispersion when parameters are estimated. *Qual Reliab Eng Int*. 2017;29(4):605-622.
20. Montgomery DC. *Introduction to Statistical Quality Control*. 7th ed. Hoboken, New Jersey: John Wiley & Sons; 2013.
21. Schoonhoven M, Riaz M, Does RJMM. Design schemes for the X¯ control chart. *Quality and Reliability Engineering International*. 2009;25(5):581-594.

22. Schoonhoven M, Riaz M, Does RJMM. Design and analysis of control charts for standard deviation with estimated parameters. *J Qual Technol*. 2011;43(4):307-333.

23. Crowder SV. Design of exponentially weighted moving average schemes. *J Qual Technol*. 1989;21(3):155-162.

24. Brook D, Evans DA. An approach to the probability distribution of cusum run-length. *Biometrika*. 1972;59(3):539-549.

25. Lucas JM, Saccucci MS. Exponentially weighted moving average control schemes: properties and enhancements. *Technometrics*. 1990;32(1):1-12.

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APPENDIX A

SIMULATION OF THE EMPIRICAL DISTRIBUTION OF $CARL_{IN}$

Step 1: Specify λ, L, m, and n
Step 2: Simulate an observation Z from the standard normal distribution
Step 3: Simulate an observation Y from the chi-square distribution with $m(n-1)$ degrees of freedom and calculate $Q=\sqrt{Y/m(n-1)}$
Step 4: Calculate p_{lk} for $l, k = -100, \ldots, 0, . \ldots, 100$, using Equation 4 and construct the matrix P
Step 5: Calculate $CARL_{IN}$ using Equation 5
Step 6: Repeat steps (1) to (5) many times (eg, 5000 times).

Order the 5000 $CARL_{IN}$ values in ascending order. This ordered set of values and their associated cumulative frequency (cumulative probability) constitute an empirical distribution.

Note that, for reasons of calculation speed and accuracy, we used $t = 201$ states, as recommended in Saleh et al.[2]

APPENDIX B

THE ALGORITHM FOR EVALUATING THE EPC PERFORMANCE OF A STANDARD PHASE II EWMA $\overline{X}$ CHART

Step 1: Fix m, n, λ, L, p, t, and ARL_0
Step 2: Generate the empirical distribution of $CARL_{IN}$ (See Appendix A)

Step 3: Calculate the 100*pth* percentile $CARL_{IN,\ p}$ of the empirical $CARL_{IN}$ distribution.

Step 4: Calculate $= \frac{CARL_{IN,p} - ARL_0}{ARL_0} \times 100$, the percentage difference between the $CARL_{IN,p}$ and the ARL_0.

Interpretation of *PD*: A negative *PD* value means that $CARL_{IN,\ p} < ARL_0$ by *PD* percentage points. A positive *PD* value means that $CARL_{IN,\ p} > ARL_0$ by *PD* percentage points.

APPENDIX C

A STEP-BY-STEP ALGORITHM FOR FINDING *L* USING THE EPC APPROACH

Step 1: Fix ε, p, ARL_0, m, n, λ, t, and a value of L in the search interval $L \in [Case\ K, \infty)$
Step 2: Generate the empirical distribution of $CARL_{IN}$(See Appendix A)
Step 3: Calculate the *pth* percentile $CARL_{IN,p}$ from the empirical distribution
Step 4: If $CARL_{IN,p} > ARL_0(1 - \varepsilon)$ stop and use the current value of L otherwise increment L and return to step 2.

To find L very quickly, for a given set of m value, start with the largest m.