

# Design and Analysis of Control Charts for Standard Deviation with Estimated Parameters

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This paper concerns the design and analysis of the standard deviation control chart with estimated limits. We consider an extensive range of statistics to estimate the in-control standard deviation (Phase I) and design the control chart for real-time process monitoring (Phase II) by determining the factors for the control limits. The Phase II performance of the design schemes is assessed when the Phase I data are uncontaminated and normally distributed as well as when the Phase I data are contaminated. We propose a robust estimation method based on the mean absolute deviation from the median supplemented with a simple screening method. It turns out that this approach is efficient under normality and performs substantially better than the traditional estimators and several robust proposals when contaminations are present.

Key Words: Average Run Length; Mean-Squared Error; Phase I; Phase II; Robust; Shewhart Control Chart; Statistical Process Control.

## Introduction

THE PERFORMANCE of a process depends on the stability of its location and dispersion parameters,

and an optimal performance requires any change in these parameters to be detected as early as possible. To monitor a process with respect to these parameters, Shewhart introduced the idea of control charts in the 1920s. The dispersion parameter of the process is controlled first, followed by the location parameter. The present paper focuses on control charts for monitoring the process standard deviation.

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Let  $Y_{ij}$ ,  $i = 1, 2, 3, \dots$  and  $j = 1, 2, \dots, n$ , denote samples of size  $n$  taken in sequence on the process variable to be monitored. We assume the  $Y_{ij}$ 's to be independent and  $N(\mu, \lambda\sigma)$  distributed, where  $\lambda$  is a constant. When  $\lambda = 1$ , the standard deviation of the process is in control; otherwise, the standard deviation has changed. Let  $\hat{\sigma}_i$  be an estimate of  $\lambda\sigma$  based on the  $i$ th sample  $Y_{ij}$ ,  $j = 1, 2, \dots, n$ . When the in-

control  $\sigma$  is known, the process standard deviation can be monitored by plotting  $\hat{\sigma}_i$  on a Shewhart-type control chart with respective upper and lower control limits

$$UCL = U_n\sigma, \quad LCL = L_n\sigma, \quad (1)$$

where  $U_n$  and  $L_n$  are factors such that, for a chosen type I error probability  $\alpha$ , we have

$$P(L_n\sigma \leq \hat{\sigma}_i \leq U_n\sigma) = 1 - \alpha.$$

When  $\hat{\sigma}_i$  falls within the control limits, the process is deemed to be in control. We define  $E_i$  as the event that  $\hat{\sigma}_i$  falls beyond the limits,  $P(E_i)$  as the probability that sample  $i$  falls beyond the limits, and RL as the run length, i.e., the number of samples until the first  $\hat{\sigma}_i$  falls beyond the limits. When  $\sigma$  is known, the events  $E_i$  are independent, and therefore RL is geometrically distributed with parameter  $p = P(E_i) = \alpha$ . It follows that the average run length (ARL) is given by  $1/p$  and that the standard deviation of the run length (SDRL) is given by  $\sqrt{1-p}/p$ .

In practice, the in-control process parameters are usually unknown. Therefore, they must be estimated from samples taken when the process is assumed to be in control. This stage in the control-charting process is called Phase I (cf., Woodall and Montgomery (1999), Vining (2009)). The monitoring stage is denoted by Phase II. Define  $\hat{\sigma}$  as an unbiased estimate of  $\sigma$  based on  $k$  samples of size  $n$ , which are denoted by  $X_{ij}$ ,  $i = 1, 2, \dots, k$ . The control limits can be estimated by

$$\widehat{UCL} = U_n\hat{\sigma}, \quad \widehat{LCL} = L_n\hat{\sigma}. \quad (2)$$

These  $U_n$  and  $L_n$  are not necessarily the same as in Equation (1) and will be different if the marginal probability of signalling is the same. Let  $F_i$  denote the event that  $\hat{\sigma}_i$  is above  $\widehat{UCL}$  or below  $\widehat{LCL}$ . We define  $P(F_i | \hat{\sigma})$  as the probability that sample  $i$  generates a signal given  $\hat{\sigma}$ , i.e.,

$$P(F_i | \hat{\sigma}) = P(\hat{\sigma}_i < \widehat{LCL} \text{ or } \hat{\sigma}_i > \widehat{UCL} | \hat{\sigma}).$$

Given  $\hat{\sigma}$ , the distribution of the run length is geometric with parameter  $P(F_i | \hat{\sigma})$ . Consequently, the conditional ARL is given by

$$E(\text{RL} | \hat{\sigma}) = \frac{1}{P(F_i | \hat{\sigma})}.$$

In contrast with the conditional RL distribution, the marginal RL distribution takes into account the random variability introduced into the charting procedure through parameter estimation. It can be obtained by averaging the conditional RL distribution

over all possible values of the parameter estimates. The unconditional  $p$  is

$$p = E(P(F_i | \hat{\sigma})),$$

the unconditional average run length is

$$\text{ARL} = E\left(\frac{1}{P(F_i | \hat{\sigma})}\right).$$

Quesenberry (1993) showed that, for the  $\bar{X}$  and  $X$  control charts, the marginal ARL is higher than in the  $\sigma$ -known case. Furthermore, a higher in-control ARL is not necessarily better because the RL distribution will reflect an increased number of short RLs as well as an increased number of long RLs. He concluded that, if limits are to behave like known limits, the number of samples ( $k$ ) in Phase I should be at least  $400/(n-1)$  for  $\bar{X}$  control charts and 300 for  $X$  control charts. Chen (1998) studied the marginal RL distribution of the standard deviation control chart under normality. He showed that, if the shift in the standard deviation in Phase II is large, the impact of parameter estimation is small. In order to achieve a performance comparable with known limits, he recommended taking at least 30 samples of size 5 and updating the limits when more samples become available. For permanent limits, at least 75 samples of size 5 should be used. Thus, the situation is somewhat better than for the  $\bar{X}$  control chart with both process mean and standard deviation estimated.

Jensen et al. (2006) conducted a literature survey of the effects of parameter estimation on control-chart properties and identified several issues for future research. One of their recommendations is to consider robust or other alternative estimators for the location and the standard deviation in Phase I applications because it seems more appropriate to use an estimator that will be robust to outliers and step changes in Phase I. Also, the effect of using these robust estimators on Phase II should be assessed (Jensen et al. (2006, p. 360)). This recommendation is the subject of the present paper, i.e., we will study alternative estimators for the standard deviation in Phase I and we will study the impact of these estimators on the Phase II performance of the standard deviation control chart.

Chen (1998) studied the standard deviation control chart when  $\sigma$  is estimated by the pooled-sample standard deviation ( $\hat{S}$ ), the mean-sample standard deviation ( $\bar{S}$ ), or the mean-sample range ( $\bar{R}$ ) under normality. He showed that the performance of the charts based on  $\hat{S}$  and  $\bar{S}$  is almost identical,

while the performance of the chart based on  $\bar{R}$  is slightly worse. Rocke (1989) proposed robust control charts based on the 25% trimmed mean of the sample ranges, the median of the sample ranges, and the mean of the sample interquartile ranges in contaminated Phase I situations. Moreover, he studied the use of a two-stage procedure whereby the initial chart is constructed first and then subgroups that seem to be out of control are excluded. Rocke (1992) gave the practical details for the construction of these charts. Wu et al. (2002) considered three alternative statistics for the sample standard deviation, namely the median of the absolute deviation from the median (MDM), the average absolute deviation from the median (ADM), and the median of the average absolute deviation (MAD), and investigated their effect on  $\bar{X}$  control-chart performance. They concluded that, if there are no or only a few contaminations in the Phase I data, ADM performs best. Otherwise, MDM is the best estimator. Riaz and Saghir (2007, 2009) showed that the statistics for the sample standard deviation based on the Gini's mean difference and the ADM are robust against nonnormality. However, they only considered the situation where a large number of samples is available in Phase I and did not consider contaminations in Phase I. Tatum (1997) clearly distinguished two types of disturbances: diffuse and localized. Diffuse disturbances are outliers that are spread over multiple samples, whereas localized disturbances affect all observations in a single sample. He proposed a method, constructed around a variant of the biweight  $A$  estimator, that is resistant to both diffuse and localized disturbances. A result of the inclusion of the biweight  $A$  method is, however, that the estimator is relatively complicated in its use. Besides several range-based methods, Tatum did not compare his method with other methods for Phase I estimation. Finally, Boyles (1997) studied the dynamic linear-model estimator for individual charts (see also Braun and Park (2008)).

In this paper, we compare an extensive number of Phase I estimators that have been presented in the literature and a number of variants on these statistics. We study their effect on the Phase II performance of the standard deviation control chart. The estimators considered are:  $\tilde{S}$ ,  $\bar{S}$ , the 25% trimmed mean of the subgroup standard deviations (rather than the 25% trimmed sample ranges because it is well known that  $S$  is more robust than  $R$ ), the mean of the subgroup standard deviations after trimming the observations in each sample, the sample interquartile range, the Gini's mean difference, the

MDM, the ADM, the MAD, and the robust estimator of Tatum (1997). Moreover, we investigate the use of a variant of the screening methods proposed by Rocke (1989) and Tatum (1997). The performance of the estimators is evaluated by assessing the mean-squared error (MSE) of the estimators under normality and in the presence of various types of contaminations. Further, we derive the constants that determine the control limits. We then have the desired marginal probability that the chart will produce a false signal in Phase II. Finally, we assess the Phase II performance of the control charts by means of a simulation study.

The paper is structured as follows. The next section introduces the estimators of the standard deviation and assesses the MSE of the estimators. Subsequently, we derive the Phase II control limits. Next, we describe the simulation procedure and the results of the simulation study. Furthermore, we discuss a real-world example implementing the various charts created. The paper ends with some concluding remarks.

## Proposed Phase I Estimators

In practice, the same statistic is generally used to estimate both the in-control standard deviation  $\sigma$  in Phase I and the standard deviation  $\lambda\sigma$  in Phase II. Because the requirements for the estimators differ between the two phases, this is not always the best choice. In Phase I, an estimator should be efficient in uncontaminated situations and robust against disturbances, whereas in Phase II the estimator should be sensitive to disturbances (cf., Jensen et al. (2006)). In this section, we present the Phase I estimators considered in our study. The first subsection introduces the estimators, while the second subsection presents the MSE of the estimators.

### Estimators of the Standard Deviation

David (1998) gave a brief account of the history of standard-deviation estimators. The traditional estimators are of course the pooled and the mean-sample standard deviation and the mean-sample range. Mahmoud et al. (2010) studied the relative efficiencies of these estimators for different sample sizes  $n$  and number of samples  $k$ . In deriving estimates of the in-control standard deviation, we will look at these as well as nine other estimators.

The first estimator of  $\sigma$  is based on the pooled-

sample standard deviation

$$\tilde{S} = \left( \frac{1}{k} \sum_{i=1}^k S_i^2 \right)^{1/2}, \tag{3}$$

where  $S_i$  is the  $i$ th sample standard deviation defined by

$$S_i = \left( \frac{1}{n-1} \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2 \right)^{1/2}.$$

The unbiased estimator is given by  $\tilde{S}/c_4(k(n-1)+1)$ , where  $c_4(m)$  is defined by

$$c_4(m) = \left( \frac{2}{m-1} \right)^{1/2} \frac{\Gamma(m/2)}{\Gamma((m-1)/2)}.$$

The second estimator is based on the mean-sample standard deviation

$$\bar{S} = \frac{1}{k} \sum_{i=1}^k S_i. \tag{4}$$

An unbiased estimator of  $\sigma$  is given by  $\bar{S}/c_4(n)$ .

Rocke (1989) proposed the trimmed mean of the sample ranges. In our study, we consider a variant of this estimator, namely, the trimmed mean of the sample standard deviations because it is well known that the standard deviation is more robust than the sample range. The trimmed mean of the sample standard deviation is given by

$$\bar{S}_a = \frac{1}{k - \lceil ka \rceil} \times \left[ \sum_{v=1}^{k - \lceil ka \rceil} \bar{S}_{(v)} \right], \tag{5}$$

where  $a$  denotes the percentage of samples to be trimmed,  $\lceil z \rceil$  denotes the smallest integer not less than  $z$  and  $\bar{S}_{(v)}$  denotes the  $v$ th ordered value of the sample standard deviations. In our study, we consider the 25% trimmed mean of the sample standard deviations. To simplify the analysis, we trim an integer number of samples. For example, the 25% trimmed mean trims off the eight largest sample standard deviations when  $k = 30$ . To provide an unbiased estimate of  $\sigma$  for the normal case, the estimate must be divided by a normalizing constant. These constants are obtained from 100,000 simulation runs. For  $n = 5$  and  $k = 20, 30, 75$ , the constants are 0.579, 0.585, and 0.568, respectively; for  $n = 9$  and  $k = 20, 30, 75$ , the constants are 0.701, 0.705, and 0.693, respectively.

Because the above estimator trims off samples instead of individual observations, we expect the estimator to be robust against localized disturbances.

We also consider a variant that is expected to be robust against diffuse disturbances, namely, the mean-sample standard deviation after trimming the observations in each sample:

$$\bar{S}'_a = \frac{1}{k} \sum_{i=1}^k S'_i, \tag{6}$$

where  $S'_i$  is the standard deviation of sample  $i$  after trimming the observations, given by

$$S'_i = \left( \frac{1}{n - 2\lceil na \rceil - 1} \sum_{v=\lceil na \rceil + 1}^{n - \lceil na \rceil} (X_{i(v)} - \bar{X}'_i)^2 \right)^{1/2},$$

where

$$\bar{X}'_i = \frac{1}{n - 2\lceil na \rceil} \sum_{v=\lceil na \rceil + 1}^{n - \lceil na \rceil} X_{i(v)},$$

with  $X_{i(v)}$  the  $v$ th ordered value in sample  $i$  and  $a$  the percentage of lowest and highest observations to be trimmed in each sample. In this study, we take 20% as our trimming percentage and, again, we trim an integer number of observations. The estimator trims off the smallest and largest observation for  $n = 5$ ; it trims off the two smallest and the two largest observations for  $n = 9$ . The normalizing constant is 0.520 for  $n = 5$  and 0.473 for  $n = 9$ .

The next estimator is based on the mean-sample range

$$\bar{R} = \frac{1}{k} \sum_{i=1}^k R_i, \tag{7}$$

where  $R_i$  is the range of the  $i$ th sample. An unbiased estimator of  $\sigma$  is  $\bar{R}/d_2(n)$ , where  $d_2(n)$  is the expected range of a random  $N(0, 1)$  sample of size  $n$ . Values of  $d_2(n)$  can be found in Duncan (1986, Table M).

The next estimator is based on the mean of the sample interquartile ranges (IQRs)

$$\overline{\text{IQR}} = \frac{1}{k} \sum_{i=1}^k \text{IQR}_i, \tag{8}$$

with  $\text{IQR}_i$  the interquartile range of sample  $i$

$$\text{IQR}_i = X_{i(n - \lceil na \rceil)} - X_{i(\lceil na \rceil + 1)}.$$

Thus, the same observations are trimmed off as in the calculation of  $S'_i$ . (Note that one would expect the IQR to correspond to  $a = 0.25$ . However, to simplify the analysis, we only trim an integer number of observations.) The normalizing constant is 0.990 for  $n = 5$  and 1.144 for  $n = 9$ .

We also consider an estimator based on Gini's mean-sample differences

$$\bar{G} = \frac{1}{k} \sum_{i=1}^k G_i, \tag{9}$$

where  $G_i$  is Gini's mean difference of sample  $i$  defined by

$$G_i = \sum_{j=1}^{n-1} \sum_{l=j+1}^n |X_{ij} - X_{il}| / (n(n-1)/2),$$

representing the mean absolute difference between any two observations in the sample. This statistic was proposed by Gini (1912), although basically the same statistic had already been proposed by Jordan (1869). An unbiased estimator of  $\sigma$  is given by  $\bar{G}/d_2(2)$ . Appendix A shows that the estimator based on Gini's mean difference can be rewritten as a linear function of order statistics and that Gini's mean difference is essentially the same as the so-called Downton estimator (Downton (1966)) and the probability-weighted moments estimator (Muhammad et al. (1993)). From David (1981, p. 191), it follows that the estimator derived from Gini's mean difference is highly efficient (98%) and is more robust to outliers than the estimators based on  $R$  or  $S$ .

An estimator of  $\sigma$  that is simpler and easier to interpret uses the mean of the sample-average absolute deviation from the median, given by

$$\overline{\text{ADM}} = \frac{1}{k} \sum_{i=1}^k \text{ADM}_i, \tag{10}$$

where  $\text{ADM}_i$  is the average absolute deviation from the median of sample  $i$ , given by

$$\text{ADM}_i = \frac{1}{n} \sum_{j=1}^n |X_{ij} - M_i|,$$

with  $M_i$  the median of sample  $i$ . An unbiased estimator of  $\sigma$  is given by  $\overline{\text{ADM}}/t_2(n)$ . Because it is difficult to obtain the constant  $t_2(n)$  analytically, it has to be obtained by simulation. Extensive tables of  $t_2(n)$  can be found in Riaz and Saghir (2009). Like  $G$ , we can rewrite  $\text{ADM}$  as a function of order statistics. As a result, we can express  $G$  in terms of  $\text{ADM}$ . The exact relationship can be found in Appendix B.

We also study the above estimator supplemented with a screening method based on control charting. Rocke (1989) proposed a two-stage procedure that first estimates  $\sigma$  by  $\bar{R}$ , then deletes any subsample that exceeds the control limits and recomputes  $\bar{R}$

using the remaining subsamples. Our approach follows a similar procedure. First, we estimate  $\sigma$  by  $\overline{\text{ADM}}$  because  $\overline{\text{ADM}}$  is expected to be more robust against outliers. For simplicity, we use for the screening method the well-known factors of the  $S/c_4(n)$  control chart corresponding to the  $3\sigma$  control limits in Phase I. Hence, the factors for the limits are 2.089 and 0 for  $n = 5$  and 1.761 and 0.239 for  $n = 9$  (cf., Table M in Duncan (1986)). Then we chart  $S/c_4(n)$ , delete any subsample that exceeds the control limits, and recompute  $\overline{\text{ADM}}$  using the remaining subsamples. We continue until all subsample estimates fall within the limits. The normalizing constant is 0.996 for  $n = 5$  and 0.998 for  $n = 9$ . The resulting estimator is denoted by  $\overline{\text{ADM}}'$ .

Next we study two other median statistics: the average of the sample medians of the absolute deviation from the median

$$\overline{\text{MDM}} = \frac{1}{k} \sum_{i=1}^k \text{MDM}_i, \tag{11}$$

with

$$\text{MDM}_i = \text{median}\{|X_{ij} - M_i|\},$$

and the mean of the sample median of the average absolute deviation

$$\overline{\text{MAD}} = \frac{1}{k} \sum_{i=1}^k \text{MAD}_i, \tag{12}$$

with

$$\text{MAD}_i = \text{median}\{|X_{ij} - \bar{X}_i|\}.$$

The normalizing constant for  $\overline{\text{MDM}}$  is 0.554 for  $n = 5$  and 0.613 for  $n = 9$ . For  $\overline{\text{MAD}}$ , the normalizing constant is 0.627 for  $n = 5$  and 0.658 for  $n = 9$ .

We also evaluate a robust estimator proposed by Tatum (1997). His method has proven to be robust to both diffuse and localized disturbances. The estimation method is constructed around a variant of the biweight  $A$  estimator. The method begins by calculating the residuals in each sample, which involves subtracting the subsample median from each value:  $\text{res}_{ij} = X_{ij} - M_i$ . If  $n$  is odd, then, in each sample, one of the residuals will be zero and is dropped. As a result, the total number of residuals is equal to  $m' = nk$  when  $n$  is even and  $m' = (n-1)k$  when  $n$  is odd. Tatum's estimator is given by

$$S_c^* = \frac{m'}{(m' - 1)^{1/2}} \times \frac{(\sum_{i=1}^k \sum_{j:|u_{ij}| < 1} \text{res}_{ij}^2 (1 - u_{ij}^2)^4)^{1/2}}{|\sum_{i=1}^k \sum_{j:|u_{ij}| < 1} (1 - u_{ij}^2)(1 - 5u_{ij}^2)|}, \tag{13}$$

TABLE 1. Normalizing Constants  $d^*(c, n, k)$  for Tatum's Estimator  $S_c^*$

$n$	$c = 7$				$c = 10$			
	$k = 20$	$k = 30$	$k = 40$	$k = 75$	$k = 20$	$k = 30$	$k = 40$	$k = 75$
5	1.070	1.069	1.068	1.068	1.054	1.053	1.053	1.052
7	1.057	1.056	1.056	1.056	1.041	1.040	1.040	1.040
9	1.052	1.051	1.050	1.050	1.034	1.034	1.033	1.033
11	1.047	1.046	1.046	1.046	1.029	1.029	1.028	1.028
13	1.044	1.044	1.043	1.043	1.026	1.025	1.025	1.025
15	1.041	1.041	1.041	1.040	1.023	1.023	1.023	1.022

where  $u_{ij} = h_i res_{ij}/(cM^*)$ ,  $M^*$  is the median of all residuals,

$$h_i = \begin{cases} 1 & E_i \leq 4.5, \\ E_i - 3.5 & 4.5 < E_i \leq 7.5, \\ c & E_i > 7.5, \end{cases}$$

and  $E_i = IQR_i/M^*$ . The constant  $c$  is a tuning constant. Each value of  $c$  leads to a different estimator. Tatum showed that  $c = 7$  gives an estimator that loses some efficiency when no disturbances are present, but gains efficiency when disturbances are present. We apply this value of  $c$  in our simulation study. Note that we have  $h(i) = E_i - 3.5$  for  $4.5 < E_i \leq 7.5$  in the equations instead of  $h(i) = E_i - 4.5$  as presented by Tatum (Tatum (1997), p. 129). This was a typographical error in the formula, resulting in too much weight on localized disturbances and thus an overestimation of  $\sigma$ . An unbiased estimator of  $\sigma$  is given by  $S_c^*/d^*(c, n, k)$ ,

where  $d^*(c, n, k)$  is the normalizing constant. During the implementation of the estimator, we discovered that, for odd values of  $n$ , the values of  $d^*(c, n, k)$  given by Table 1 in Tatum (1997) should be adapted. We use the corrected values, which are presented in Table 1 below. The resulting estimator is denoted by  $D7$  as in Tatum (1997).

The estimators considered are summarized in Table 2.  $S_I$  denotes the estimator used in Phase I to estimate the in-control  $\sigma$ .

**Efficiency of Proposed Estimators**

For comparison purposes, we assess the MSE of the proposed Phase I estimators, as was done in Tatum (1997). The MSE will be estimated as

$$MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\sigma}^i - \sigma)^2, \tag{14}$$

TABLE 2. Proposed Estimators for the Standard Deviation

$S_I$	Description
$\tilde{S}$	Pooled sample standard deviation
$\bar{S}$	Mean of sample standard deviations
$\bar{S}_{25}$	25% trimmed mean of sample standard deviations
$\bar{S}_{20}$	Mean of sample standard deviations after trimming the sample observations
$\bar{R}$	Mean of sample ranges
$\overline{IQR}$	Mean of sample interquartile ranges
$\bar{G}$	Mean of the sample Gini's mean differences
$\overline{ADM}$	Mean of sample averages of absolute deviation from the median
$\overline{ADM}'$	$\overline{ADM}$ after subgroup screening
$\overline{MDM}$	Mean of the sample medians of the absolute deviation from the median
$\overline{MAD}$	Mean of sample medians of the absolute deviation from the mean
$D7$	Tatum's robust estimator

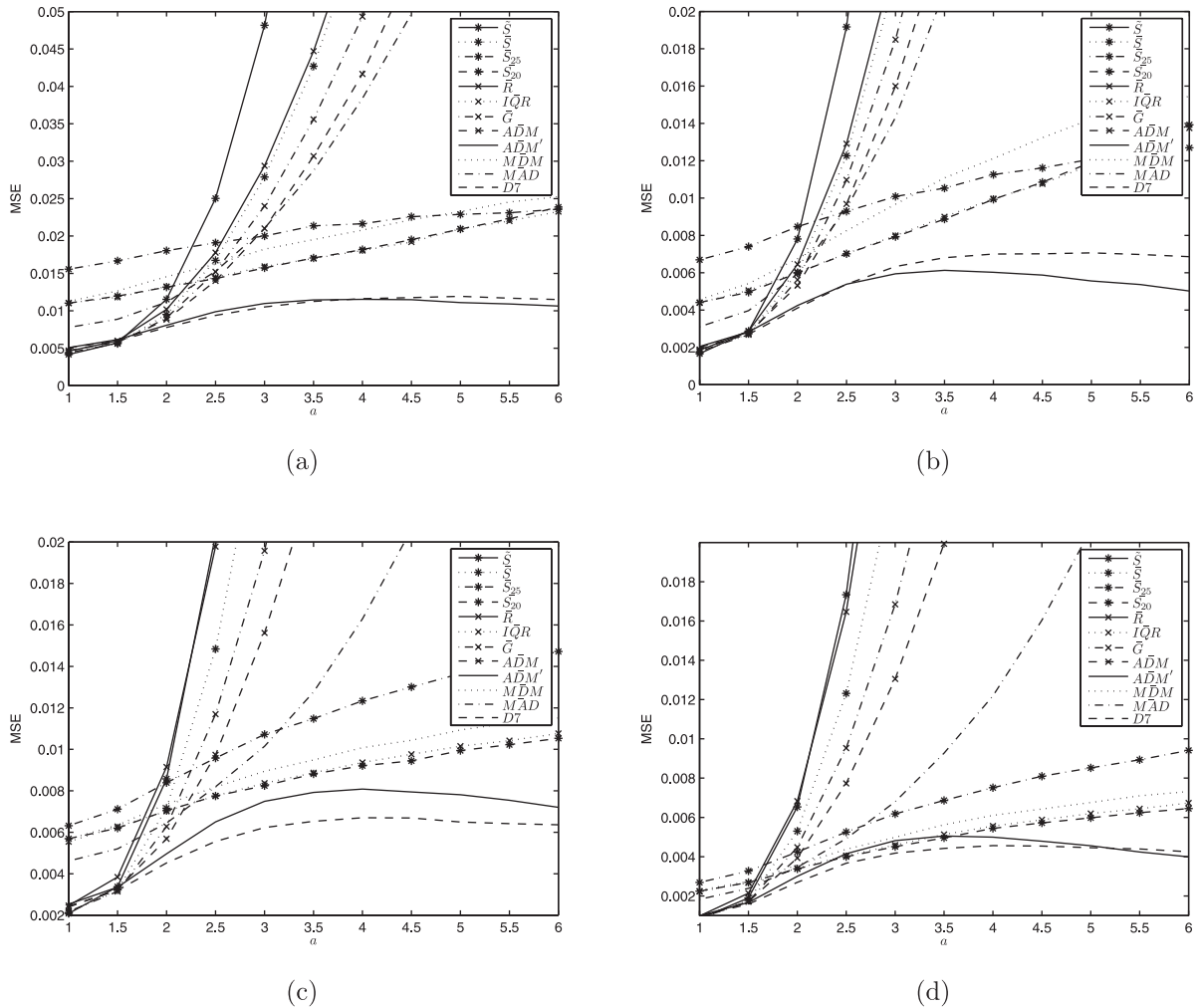


FIGURE 1. MSE of Estimators when Symmetric Diffuse Variance Disturbances Are Present. (a)  $n = 5, k = 30$ ; (b)  $n = 5, k = 75$ ; (c)  $n = 9, k = 30$ ; (d)  $n = 9, k = 75$ .

where  $\hat{\sigma}^i$  is the value of the unbiased estimate in the  $i$ th simulation run (note that  $\hat{\sigma}^i$  differs from  $\hat{\sigma}_i$ , the latter denoting the Phase II estimates of the standard deviation) and  $N$  is the number of simulation runs. We include the uncontaminated case, i.e., the situation where all  $X_{ij}$  are from the  $N(0, 1)$  distribution as well as four types of disturbances (cf. Tatum (1997)):

1. A model for diffuse symmetric variance disturbances in which each observation has a 95% probability of being drawn from the  $N(0, 1)$  distribution and a 5% probability of being drawn from the  $N(0, a)$  distribution, with  $a = 1.5, 2.0, \dots, 5.5, 6.0$ .
2. A model for diffuse asymmetric variance dis-

turbances in which each observation is drawn from the  $N(0, 1)$  distribution and has a 5% probability of having a multiple of a  $\chi^2_1$  variable added to it, with the multiplier equal to 0.5, 1.0, ..., 4.5, 5.0.

3. A model for localized variance disturbances in which observations in three (when  $k = 30$ ) or six (when  $k = 75$ ) samples are drawn from the  $N(0, a)$  distribution, with  $a = 1.5, 2.0, \dots, 5.5, 6.0$ .
4. A model for diffuse mean disturbances in which each observation has a 95% probability of being drawn from the  $N(0, 1)$  distribution and a 5% probability of being drawn from the  $N(b, 1)$  distribution, with  $b = 0.5, 1.0, \dots, 9.0, 9.5$ .

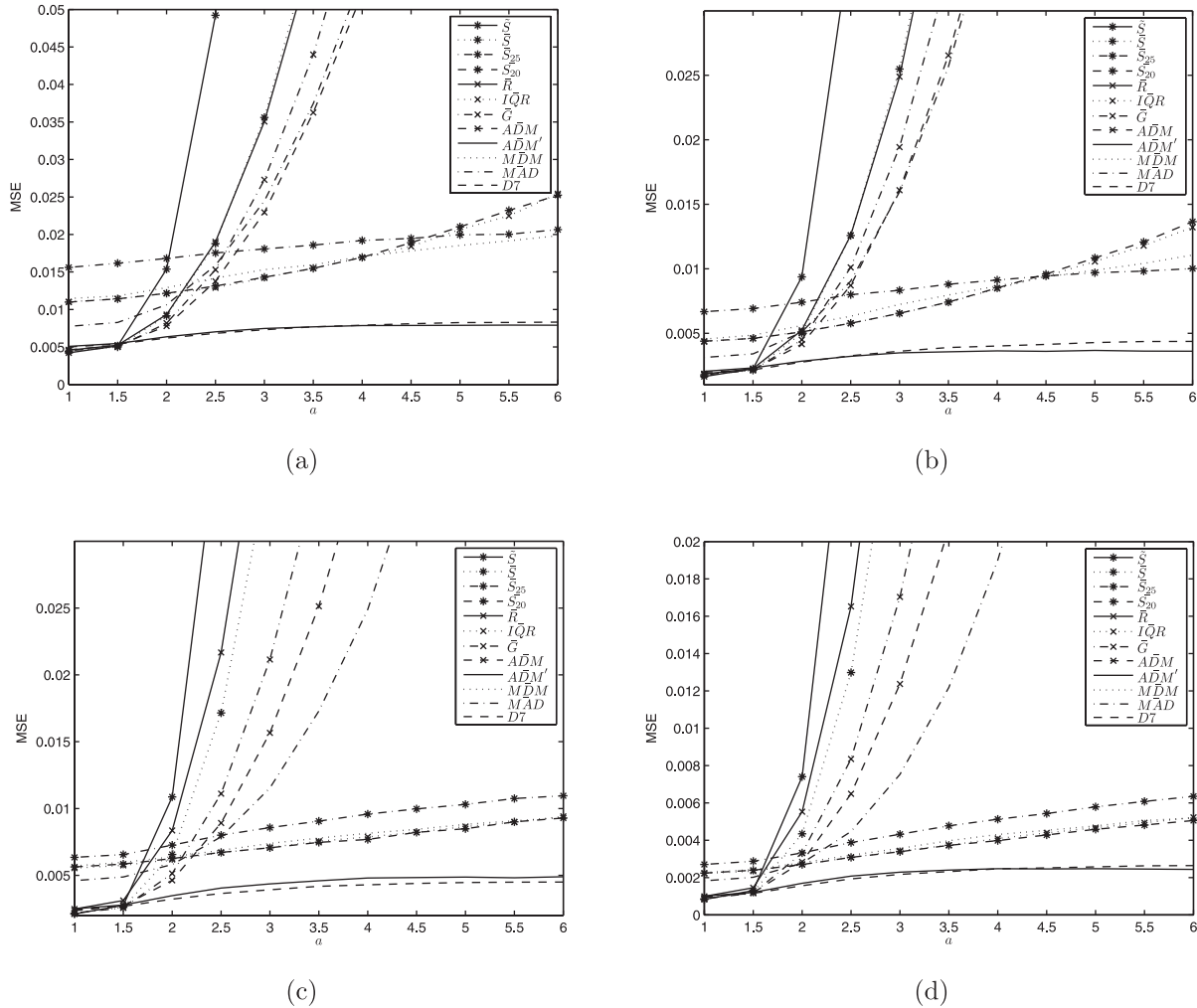


FIGURE 2. MSE of Estimators when Asymmetric Diffuse Variance Disturbances Are Present. (a)  $n = 5, k = 30$ ; (b)  $n = 5, k = 75$ ; (c)  $n = 9, k = 30$ ; (d)  $n = 9, k = 75$ .

The MSE is obtained for  $k = 30, 75$  subgroups of sizes  $n = 5, 9$ . The number of simulation runs  $N$  is equal to 50,000. (Note that Tatum (1997) used 10,000 simulation runs.)

The following results can be observed (see Figures 1–4). When no contaminations are present,  $\bar{S}_{25}$ ,  $\bar{MDM}$ ,  $\bar{S}_{20}$ ,  $\bar{IQR}$ , and  $\bar{MAD}$  are less efficient than any of the other estimators because they use less information. The efficiency of the other estimators is almost similar when no contaminations are present.

When symmetric diffuse variance disturbances are present (Figure 1), the best performing estimators are  $\bar{D7}$  and  $\bar{ADM}'$ . The fact that the performance of  $\bar{ADM}'$  is similar to  $\bar{D7}$  is interesting because the former is more intuitive and the estimates are simpler

to obtain. Tatum (1997) showed that the screening procedure based on the chart with  $\sigma$  estimated by  $\bar{R}$  fails to match  $\bar{D7}$  in this situation, which is due to the fact that  $\bar{R}$  is more sensitive to outliers. Thus, using a robust statistic like  $\bar{ADM}$ , supplemented with subgroup screening by means of the control chart (resulting in  $\bar{ADM}'$ ), works very well when symmetric diffuse outliers are present. The estimators  $\bar{S}_{25}$ ,  $\bar{S}_{20}$ ,  $\bar{IQR}$ , and  $\bar{MDM}$  are more robust than the traditional estimators but less robust than  $\bar{D7}$  and  $\bar{ADM}'$ . Another result worth noting is that  $\bar{S}$  performs worst in this situation (with comparable bad performance like  $\bar{S}$  and  $\bar{R}$ ). While others (e.g., Mahmoud et al. (2010)) recommend using this estimator because it is most efficient in the absence of contaminations, we see that its performance decreases most quickly



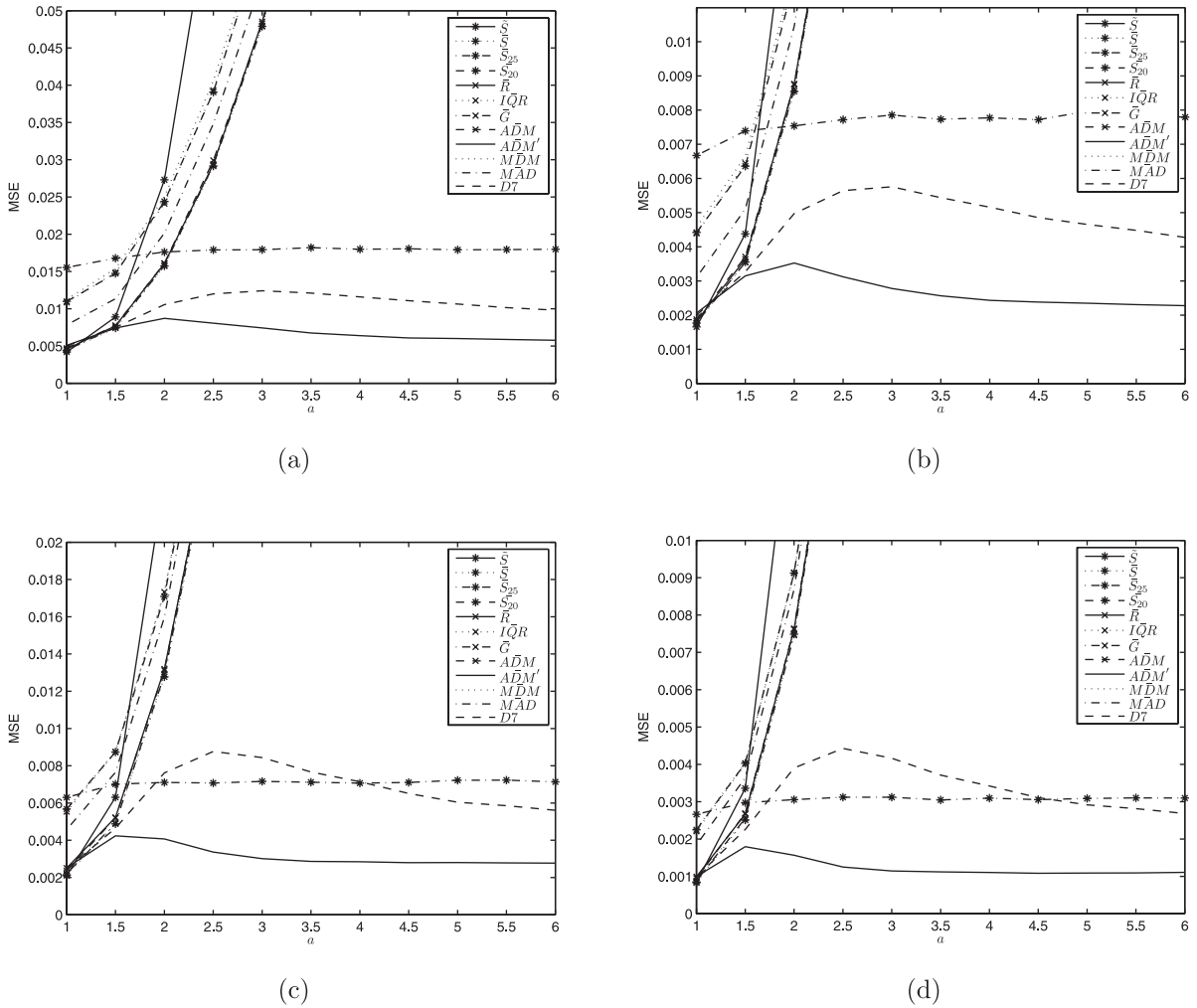


FIGURE 3. MSE of Estimators when Localized Diffuse Variance Disturbances Are Present. (a)  $n = 5, k = 30$ ; (b)  $n = 5, k = 75$ ; (c)  $n = 9, k = 30$ ; (d)  $n = 9, k = 75$ .

when there are outliers. The estimators  $\bar{G}$  and  $\overline{ADM}$  are efficient when no contaminations are present and perform better than the traditional estimators ( $\hat{S}, \bar{S}$ , and  $\bar{R}$ ) in the case of occasional outliers. The effect is more pronounced for  $n = 9$  than for  $n = 5$ .

When asymmetric diffuse variance disturbances are present (Figure 2), the same general results are found as for symmetric diffuse variance disturbances. Tatum (1997) showed that, when  $n = 9$ ,  $D7$  is superior to several other estimators, including the estimator resulting from subgroup screening based on  $\bar{R}$ . Our subgroup screening algorithm produces outcomes similar to Tatum’s estimator. Note that, to estimate  $\sigma$ , we use an estimator that is less sensitive to outliers, namely  $\overline{ADM}$  rather than  $\bar{R}$ .

In the case of localized variance disturbances (Figure 3), the estimator that performs best is  $\overline{ADM}'$ , followed by  $D7$  and then by  $\bar{S}_{25}$ . It is interesting to see that  $\overline{ADM}'$  performs substantially better than  $D7$ . In other words, screening based on the control-charting procedure in Phase I seems more effective than using  $D7$  when the data are contaminated by localized variance disturbances.

When diffuse mean disturbances are present in Phase I (Figure 4),  $D7$  performs best, followed by  $\overline{ADM}'$ . The differences appear primarily for  $n = 9$ . When there is a possibility of this type of outliers in practice, we recommend using  $D7$  or screening on the basis of an individual chart. The latter is a subject for future research.

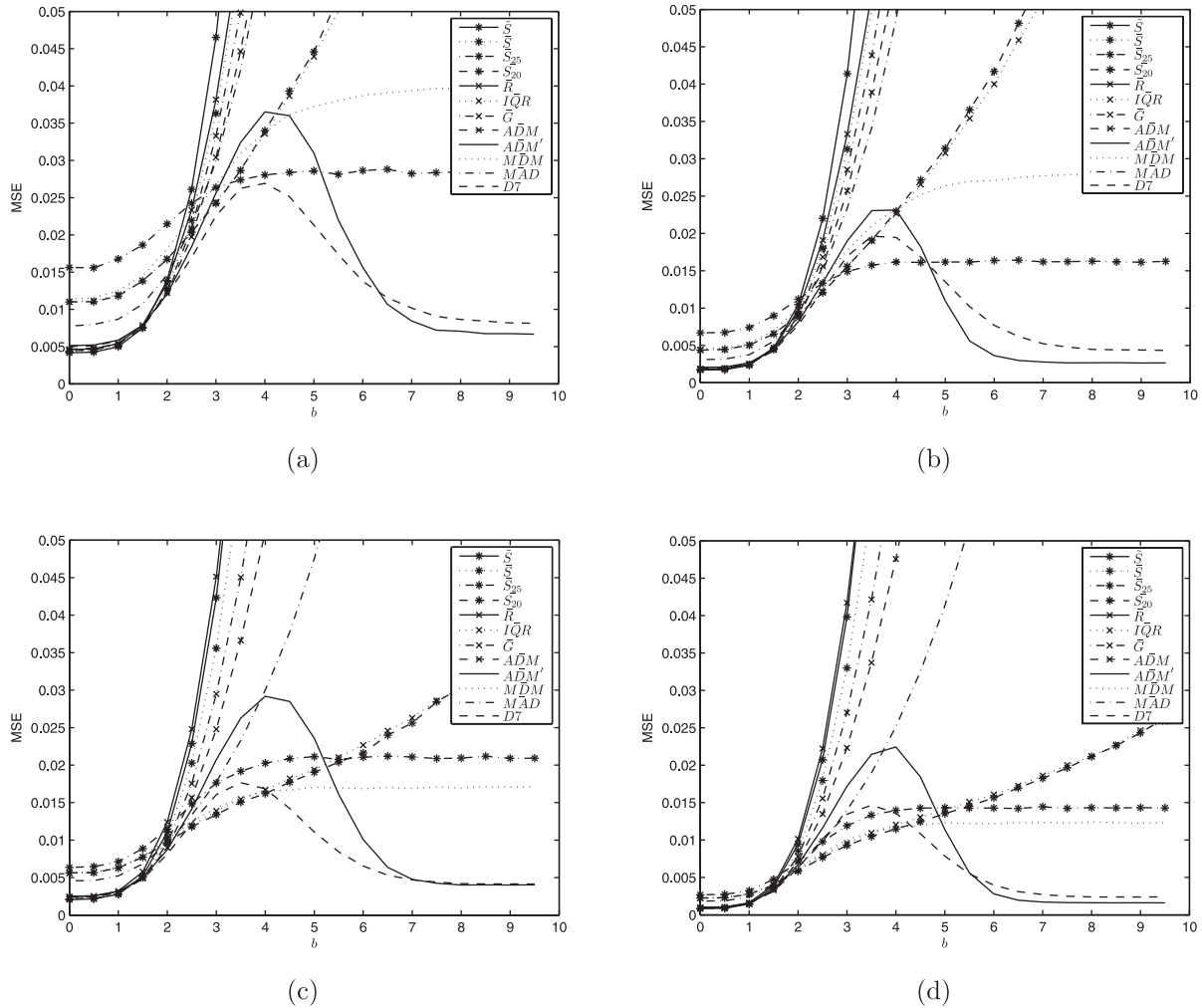


FIGURE 4. MSE of Estimators when Diffuse Mean Disturbances Are Present. (a)  $n = 5, k = 30$ ; (b)  $n = 5, k = 75$ ; (c)  $n = 9, k = 30$ ; (d)  $n = 9, k = 75$ .

To summarize, the most efficient estimators are  $D7$  and  $\overline{ADM}$  when there are diffuse variance disturbances,  $\overline{ADM}$  when there are localized variance disturbances, and  $D7$  when there are mean-shift disturbances.

### Derivation of the Phase II Control Limits

The design of the Phase II control charts requires a derivation of the factors  $U_n$  and  $L_n$  in Equation (2) to control the unconditional in-control  $p$ . Hillier (1969) showed for the  $R$  chart that, when the limits are estimated, the factors  $U_n$  and  $L_n$  derived for the  $\sigma$ -known case will not produce the desired signaling probability. To address this issue, he derived the fac-

tors based on  $n, k$ , and  $\alpha$  for the  $R$ -chart in such a way that  $p$  equals  $\alpha$ . Yang and Hillier (1970) derived correction factors for the  $S$  and  $\tilde{S}$  charts. The solution suggested by Hillier (1969) is well known as a solution for short production runs. Another advantage of designing based on the marginal  $p$  is that it seems more tractable because of the dependence in Phase II due to the estimated  $\hat{\sigma}$ . On the other hand, the ARL gives an indication of the expected run length and so is intuitively very appealing. The disadvantage of the ARL is, however, that it is determined by the occurrence of extremely long runs while, in practice, processes do not remain unchanged for a very long period (see also Does and Schriever (1992)). Nedumaran and Pignatiello (2001) developed an approach for constructing  $\bar{X}$  control limits that attempt

TABLE 3. Factors  $U_n$  and  $L_n$  to Determine Phase II Control Limits

		Factors for Control Limits											
		$n = 5$						$n = 9$					
		$k = 20$		$k = 30$		$k = 75$		$k = 20$		$k = 30$		$k = 75$	
		$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$
$\tilde{S}$	Eq. (15)	2.352	0.171	2.315	0.172	2.272	0.173	1.890	0.349	1.872	0.350	1.851	0.351
$\bar{S}$	Eq. (18)	2.357	0.171	2.318	0.172	2.274	0.173	1.892	0.349	1.873	0.350	1.852	0.351
$\bar{S}_{25}$	Eq. (18)	2.704	0.167	2.527	0.169	2.359	0.171	2.011	0.342	1.946	0.345	1.883	0.349
$\bar{S}_{20}$	Eq. (18)	2.540	0.169	2.438	0.170	2.319	0.172	1.987	0.343	1.934	0.346	1.876	0.349
$\bar{R}$	Eq. (18)	2.364	0.171	2.322	0.172	2.275	0.173	1.900	0.348	1.879	0.349	1.854	0.351
$\overline{IQR}$	Eq. (18)	2.541	0.169	2.439	0.170	2.318	0.172	1.982	0.343	1.933	0.346	1.875	0.350
$\bar{G}$	Eq. (18)	2.359	0.171	2.320	0.172	2.275	0.173	1.894	0.348	1.874	0.350	1.852	0.351
$\overline{ADM}$	Eq. (18)	2.366	0.171	2.324	0.172	2.275	0.173	1.898	0.348	1.877	0.349	1.854	0.351
$\overline{ADM}'$	Eq. (18)	2.376	0.171	2.332	0.171	2.279	0.172	1.901	0.348	1.879	0.349	1.854	0.351
$\overline{MDM}$	Eq. (18)	2.554	0.169	2.442	0.170	2.322	0.172	1.987	0.343	1.936	0.346	1.876	0.349
$\overline{MAD}$	Eq. (18)	2.447	0.170	2.380	0.171	2.296	0.172	1.956	0.345	1.915	0.347	1.868	0.350
$D7$	Eq. (18)	2.376	0.171	2.331	0.172	2.278	0.172	1.901	0.348	1.879	0.349	1.854	0.351

to match any percentile point of the run-length distribution.

In this study, we derive the factors  $U_n$  and  $L_n$  to obtain the desired value for  $p$ . Later we will show that this issue is less important for the standard-deviation control chart than for the  $\bar{X}$ - and  $X$ -charts, because the estimation effect is less pronounced for the standard-deviation control chart.

The factors  $U_n$  and  $L_n$  depend on  $n$ ,  $k$ , and  $\alpha$ . The Phase I estimators considered are the estimators presented in Table 2. We employ the same statistic, namely  $S/c_4(n)$ , as the Phase II charting statistic in each case so that any differences between the charts are entirely due to differences introduced by the Phase I estimators. Below, we present the derivation of the factors for these charts.

We start with the factors for the chart where  $\hat{\sigma}$  is estimated by  $\tilde{S}/c_4(k(n-1)+1)$  (see Equation (3)). Exact results for this chart can be calculated and can also be found in Yang and Hillier (1970). We derive the factor for the upper control limit; the factor for the lower control limit can be obtained in a similar way. Note that  $S_i$  and  $\tilde{S}$  are independent, so the factors can be chosen as the upper and lower  $\alpha/2$  quantiles of the distribution  $S_i/\tilde{S}$ . We can write  $(S_i/\tilde{S})^2$

as

$$\frac{(n-1)S_i^2/\sigma^2}{k(n-1)\tilde{S}^2/\sigma^2} \cdot \frac{1/(n-1)}{1/k(n-1)},$$

which is distributed as

$$\frac{\chi_{n-1}^2/(n-1)}{\chi_{k(n-1)}^2/(k(n-1))} = F_{n-1,k(n-1)},$$

where  $\chi_m^2$  denotes a chi-square distribution with  $m$  degrees of freedom and  $F_{v,w}$  denotes an  $F$ -distribution with  $v$  numerator degrees of freedom and  $w$  denominator degrees of freedom. Hence,

$$U_n = \frac{\sqrt{F_{n-1,k(n-1)}(1-\alpha/2)}c_4(k(n-1)+1)}{c_4(n)}. \tag{15}$$

For the charts based on the other Phase I estimators, we use the result of Patnaik (1950). Patnaik approximates the distribution of  $\bar{R}/\sigma$  by  $a(n,k)\chi_{\nu(n,k)}/\sqrt{\nu(n,k)}$ , where  $\chi_{\nu(n,k)}$  is the square root of a chi-square distribution with  $\nu(n,k)$  degrees of freedom and  $a(n,k)$  is a scale factor. The factors  $a(n,k)$  and  $\nu(n,k)$  are obtained by equating the first two moments of  $\bar{R}/\sigma$  to the first two moments of  $a(n,k)\chi_{\nu(n,k)}/\sqrt{\nu(n,k)}$ . Patnaik's approach can also be applied to approximate the distribution of  $\hat{\sigma}/\sigma$ , where  $\hat{\sigma}$  is obtained via one of the unbiased estimators of the standard deviation in Phase I. Let

TABLE 4. In-Control  $p \times 10^2$  of Control Limits. The estimated relative standard error is never worse than 1%.

		$p \times 10^2$											
		$n = 5$						$n = 9$					
		$k = 20$		$k = 30$		$k = 75$		$k = 20$		$k = 30$		$k = 75$	
$S_I$		$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$	$U_n$	$L_n$
$\tilde{S}$	Eq. (15)	0.135	0.135	0.135	0.135	0.135	0.135	0.134	0.135	0.134	0.135	0.134	0.135
$\bar{S}$	Eq. (18)	0.135	0.135	0.135	0.135	0.135	0.135	0.136	0.134	0.136	0.136	0.134	0.135
$\bar{S}_{25}$	Eq. (18)	0.131	0.134	0.135	0.135	0.135	0.134	0.141	0.135	0.137	0.134	0.134	0.135
$\bar{S}_{20}$	Eq. (18)	0.135	0.135	0.135	0.134	0.136	0.135	0.131	0.134	0.135	0.135	0.135	0.133
$\bar{R}$	Eq. (18)	0.135	0.134	0.135	0.136	0.135	0.137	0.135	0.135	0.134	0.134	0.134	0.136
$\overline{IQR}$	Eq. (18)	0.130	0.134	0.131	0.134	0.135	0.135	0.134	0.134	0.133	0.135	0.133	0.137
$\bar{G}$	Eq. (18)	0.133	0.134	0.136	0.135	0.134	0.136	0.133	0.134	0.134	0.136	0.134	0.136
$\overline{ADM}$	Eq. (18)	0.132	0.134	0.135	0.135	0.134	0.137	0.134	0.135	0.134	0.134	0.133	0.136
$\overline{ADM}'$	Eq. (18)	0.136	0.135	0.137	0.133	0.134	0.134	0.134	0.135	0.138	0.134	0.135	0.135
$\overline{MDM}$	Eq. (18)	0.130	0.136	0.133	0.134	0.134	0.136	0.134	0.134	0.138	0.134	0.136	0.133
$\overline{MAD}$	Eq. (18)	0.131	0.135	0.130	0.135	0.134	0.135	0.134	0.136	0.133	0.135	0.133	0.136
$D7$	Eq. (18)	0.135	0.135	0.134	0.136	0.136	0.133	0.134	0.135	0.136	0.134	0.133	0.136

$M_1 = E(\hat{\sigma}/\sigma) = 1$  and  $M_2 = \text{Var}(\hat{\sigma}/\sigma)$ . From Patnaik (1950), it follows that the values of  $\nu(n, k)$  and  $a(n, k)$  are

$$\nu(n, k) = 1/(-2 + 2\sqrt{1 + 2M_2 + 1/(16\nu(n, k))^3}), \tag{16}$$

$$a(n, k) = 1 + \frac{1}{4\nu(n, k)} + \frac{1}{32\nu^2(n, k)} - \frac{5}{128\nu^3(n, k)}. \tag{17}$$

Because

$$\frac{(S_i/\sigma)^2}{(c_4(n)\hat{\sigma}/\sigma)^2}$$

is distributed as

$$\frac{\chi_{n-1}^2/(n-1)}{c_4^2(n)a^2(n,k)\chi_{\nu(n,k)}^2/\nu(n,k)} = \frac{F_{n-1,\nu(n,k)}}{c_4^2(n)a^2(n,k)},$$

it follows that

$$U_n = \sqrt{F_{(n-1),\nu(n,k)}(1 - \alpha/2)/(c_4(n)a(n, k))}. \tag{18}$$

In Table 3, we summarize the factors  $U_n$  and  $L_n$  for the control charts with  $k = 20, 30, 75$  subgroups of sizes  $n = 5, 9$  and  $\alpha = 0.0027$ . For other situations, values of  $M_2$  can be derived by simulating  $\hat{\sigma}/\sigma$ . Then the constants  $\nu(n, k)$  and  $a(n, k)$  can be readily obtained from Equations (16) and (17).

To judge the quality of the proposed corrections, we evaluate the marginal probabilities of a false signal ( $p$ ) in Phase II. The probabilities, presented in Table 4, are assessed using 50,000 simulation runs. This is enough to obtain a sufficiently small relative standard error.

### Control-Chart Performance

In this section, we evaluate the performance of the design schemes presented above. The design schemes are set up in the uncontaminated normal situation and several contaminated situations. We consider models similar to those used to assess the MSE with  $a, b$  and the multiplier equal to 4 to simulate the contaminated case (cf., subsection entitled Efficiency of Proposed Estimators).

The performance of the design schemes is assessed in terms of the unconditional  $p$  and ARL as well as the conditional ARL associated with the 2.5% and 97.5% quantiles of the distribution of  $\hat{\sigma}$ . We consider different shifts in the standard deviation  $\lambda\sigma$  in Phase II, namely,  $\lambda$  equal to 0.5, 1, 1.5, and 2. The performance characteristics are obtained by simulation. The next section describes the simulation method, followed by the results for the control charts constructed in the uncontaminated situation and various contaminated situations.

### Simulation Procedure

The unconditional  $p$  and ARL for estimated control limits are determined by averaging the conditional characteristics, i.e., the characteristics for a given set of estimated control limits, over all values of the control limits produced in the Phase I simulation runs: Let  $X_{ij}$ ,  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, n$ , denote the Phase I data and let  $Y_{ij}$ ,  $i = 1, 2, \dots$  and  $j = 1, 2, \dots, n$ , denote the Phase II data. For each Phase I dataset of  $k$  samples of size  $n$ , we determine the estimate of the standard deviation  $\hat{\sigma}$  and the control limits  $\widehat{UCL}$  and  $\widehat{LCL}$ . Let  $S_i/c_4(n)$  be an estimate of  $\lambda\sigma$  based on the  $i$ th sample  $Y_{ij}$ ,  $j = 1, 2, \dots, n$ . Further, let  $F_i$  denote the event that  $S_i/c_4(n)$  is above  $\widehat{UCL}$  or below  $\widehat{LCL}$ . We define  $P(F_i | \hat{\sigma})$  as the probability that sample  $i$  generates a signal given  $\hat{\sigma}$ , i.e.,

$$P(F_i | \hat{\sigma}) = P(S_i/c_4(n) < \widehat{LCL} \text{ or } S_i/c_4(n) > \widehat{UCL} | \hat{\sigma}).$$

Given  $\hat{\sigma}$ , the distribution of the run length is geometric with parameter  $P(F_i | \hat{\sigma})$ . Consequently, the conditional ARL is given by

$$E(\text{RL} | \hat{\sigma}) = \frac{1}{P(F_i | \hat{\sigma})}.$$

When we take the expectation over the  $X_{ij}$ 's, we get the unconditional probability of a signal

$$P = E(P(F_i | \hat{\sigma})),$$

and the unconditional average run length

$$\text{ARL} = E(E(\text{RL} | \hat{\sigma})).$$

These expectations are obtained by simulation: 50,000 datasets are generated and, for each dataset,  $P(F_i | \hat{\sigma})$  and  $E(\text{RL} | \hat{\sigma})$  are computed. By averaging these values, we obtain the unconditional values. We also present the conditional ARL values associated with the 2.5% and 97.5% quantiles of the distribution of  $\hat{\sigma}$ .

### Simulation Results

First we consider the situation where the process follows a normal distribution and the Phase I data are not contaminated. We investigate the impact of the estimator used to estimate  $\sigma$  in Phase I. Tables 5 and 6 present the marginal probability of one sample generating a signal ( $p$ ), the marginal average run length (ARL), and the upper and lower conditional ARL values corresponding to the upper and lower 0.025 quantiles of the distribution of  $\hat{\sigma}$ . When  $\lambda = 1$ , the process is in control, so we want  $p$  to be as low

as possible and ARL to be as high as possible. When  $\lambda \neq 1$ , i.e., in the out-of-control situation, we want to achieve the opposite. The tables show that, when the limits are estimated, the in-control ARL is higher than the desired 370 (the control limits are chosen to provide an unconditional  $p$  of 0.0027), the value that is achieved when the limits are known. Note that the increase in the marginal ARL due to the estimation process is not as large as for the  $\bar{X}$  control chart. The reason is that, for the  $\bar{X}$  control chart, the run-length distribution is very right skewed, which would give a very large unconditional ARL. This seems to be less the case for standard-deviation control charts.

We also study the conditional ARL values (or, equivalently, the conditional  $p$  values, because the conditional RL distribution is simply geometric with parameter equal to the conditional  $p$ ). The first value in parentheses represents the ARL for the 2.5% quantile of the distribution of  $\hat{\sigma}$ , while the second value represents the ARL for the 97.5% quantile of the distribution of  $\hat{\sigma}$ . The results show that the conditional ARL values vary quite strongly, even when  $k$  equals 75. When  $\lambda$  equals 0.5, we see that a lower value of  $\hat{\sigma}$  gives a higher ARL and vice versa. The reason is that a smaller value of  $\hat{\sigma}$  in Phase I results in a lower value for the lower control limit and hence a lower probability of detecting a decrease in the standard deviation in Phase II. In the normal uncontaminated situation, we observe a nice pattern for all the estimators: the upper and lower conditional ARL values in the in-control situation are higher than in the out-of-control situation. However, this is not always the case when there are contaminations in Phase I (Tables 7–14). Confining ourselves to the conditional ARL values in the contaminated case, we judge the upper and lower conditional ARL values as good, provided that they do not change too much from the values observed in the uncontaminated normal case.

When we compare the differences between the estimators in the situation where the Phase I data are uncontaminated (Tables 5 and 6),  $\tilde{S}$ ,  $\bar{S}$ ,  $\bar{R}$ ,  $\bar{G}$ ,  $\overline{ADM}$ ,  $\overline{ADM}'$ , and  $D7$  produce very similar outcomes. The estimators  $\bar{S}_{25}$ ,  $\bar{S}_{20}$ ,  $\overline{IQR}$ ,  $\overline{MDM}$ , and  $\overline{MAD}$  are less powerful under normality.

The performance of the charts in the case of contaminated data are tabulated in Tables 7–14. The same general results are found as for the MSE comparisons. The most important points are:

- The chart based on  $\tilde{S}$  is most powerful under normality; however, its performance decreases

TABLE 5. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values Under Normality for  $n = 5$ 

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.019	0.0027	0.084	0.32	54.7 (86.7; 33.7)	418 (151; 455)	14.5 (5.94; 33.0)	3.28 (2.18; 5.10)
	$\bar{S}$	0.019	0.0027	0.083	0.32	54.7 (87.8; 33.4)	419 (150; 451)	14.8 (5.90; 34.2)	3.30 (2.18; 5.18)
	$\bar{S}_{25}$	0.019	0.0027	0.058	0.25	65.1 (155; 24.7)	535 (91.9; 334)	47.8 (4.78; 248)	5.45 (1.95; 15.2)
	$\bar{S}_{20}$	0.019	0.0027	0.068	0.27	60.9 (126; 27.1)	490 (104; 369)	28.3 (5.03; 113)	4.29 (2.01; 9.63)
	$\bar{R}$	0.019	0.0027	0.082	0.32	54.9 (88.8; 32.9)	421 (147; 447)	15.1 (5.85; 35.9)	3.33 (2.17; 5.31)
	$\overline{\text{IQR}}$	0.019	0.0026	0.067	0.27	61.0 (125; 27.0)	490 (106; 367)	28.5 (5.06; 114)	4.32 (2.01; 9.66)
	$\bar{G}$	0.019	0.0027	0.083	0.32	54.8 (88.3; 33.2)	421 (148; 450)	14.9 (5.87; 35.1)	3.52 (2.17; 5.53)
	$\overline{\text{ADM}}$	0.020	0.0027	0.082	0.32	54.8 (89.1; 32.8)	423 (148; 444)	15.2 (5.87; 36.6)	3.34 (2.16; 5.34)
	$\overline{\text{ADM}}'$	0.019	0.0027	0.081	0.31	56.5 (95.2; 33.2)	434 (138; 451)	15.7 (5.69; 39.3)	3.39 (2.13; 5.50)
	$\overline{\text{MDM}}$	0.019	0.0027	0.067	0.27	60.9 (129; 26.6)	490 (101; 365)	29.2 (5.02; 123)	4.37 (2.01; 10.0)
	$\overline{\text{MAD}}$	0.019	0.0026	0.074	0.29	57.7 (107; 29.4)	457 (124; 404)	20.4 (5.40; 65.6)	3.77 (2.08; 7.20)
	$D7$	0.020	0.0027	0.081	0.31	55.1 (92.0; 32.4)	427 (140; 442)	15.7 (5.72; 38.7)	3.38 (2.14; 5.49)
	75	$\tilde{S}$	0.019	0.0027	0.090	0.34	52.7 (70.8; 38.8)	391 (208; 479)	11.9 (6.98; 19.8)
$\bar{S}$		0.020	0.0027	0.090	0.34	52.7 (71.2; 38.6)	392 (205; 478)	12.0 (6.94; 20.1)	3.03 (2.35; 3.94)
$\bar{S}_{25}$		0.019	0.0027	0.077	0.30	57.1 (101; 30.8)	446 (127; 423)	18.2 (5.24; 52.5)	3.60 (2.10; 6.44)
$\bar{S}_{20}$		0.019	0.0027	0.083	0.32	54.8 (88.2; 33.2)	420 (150; 451)	14.8 (5.90; 34.7)	3.30 (2.17; 5.20)
$\bar{R}$		0.020	0.0027	0.090	0.34	52.3 (71.1; 38.0)	391 (203; 475)	12.1 (6.88; 20.6)	3.05 (2.34; 4.00)
$\overline{\text{IQR}}$		0.020	0.0027	0.083	0.32	54.7 (88.0; 33.1)	419 (150; 450)	14.8 (5.90; 34.6)	3.30 (2.17; 5.20)
$\bar{G}$		0.020	0.0027	0.090	0.34	52.2 (70.8; 38.1)	391 (206; 476)	12.0 (6.93; 20.4)	3.04 (2.35; 3.98)
$\overline{\text{ADM}}$		0.020	0.0027	0.090	0.34	52.2 (71.4; 37.8)	391 (201; 474)	12.1 (6.87; 20.7)	3.04 (2.34; 4.00)
$\overline{\text{ADM}}'$		0.019	0.0027	0.089	0.33	53.4 (74.4; 38.2)	399 (194; 484)	12.3 (6.74; 21.6)	3.07 (2.32; 4.09)
$\overline{\text{MDM}}$		0.020	0.0027	0.082	0.32	54.7 (88.0; 32.8)	422 (150; 449)	15.0 (5.89; 36.1)	3.33 (2.17; 5.29)
$\overline{\text{MAD}}$		0.019	0.0027	0.086	0.33	53.9 (80.2; 35.4)	409 (174; 471)	13.3 (6.29; 27.2)	3.17 (2.25; 4.59)
$D7$		0.019	0.0027	0.090	0.34	53.6 (74.1; 38.5)	396 (197; 485)	12.1 (6.75; 21.1)	3.04 (2.32; 4.06)

TABLE 6. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values Under Normality for  $n = 9$

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.12	0.0027	0.17	0.58	9.05 (14.4; 5.64)	402 (173; 394)	6.48 (3.52; 11.9)	1.74 (1.42; 2.22)
	$\bar{S}$	0.12	0.0027	0.17	0.58	9.02 (14.4; 5.57)	400 (172; 390)	6.52 (3.50; 12.1)	1.75 (1.42; 2.23)
	$\bar{S}_{25}$	0.11	0.0027	0.14	0.52	10.6 (24.2; 4.52)	479 (108; 296)	10.4 (2.97; 32.3)	2.02 (1.34; 3.35)
	$\bar{S}_{20}$	0.11	0.0027	0.15	0.53	10.3 (22.3; 4.59)	462 (119; 306)	9.60 (3.05; 28.4)	1.96 (1.35; 3.15)
	$\bar{R}$	0.12	0.0027	0.17	0.58	9.23 (15.1; 5.48)	410 (163; 383)	6.74 (3.44; 13.3)	1.77 (1.42; 2.31)
	$\overline{IQR}$	0.11	0.0027	0.15	0.54	10.3 (22.0; 4.59)	463 (118; 306)	9.52 (3.06; 28.0)	1.96 (1.35; 3.13)
	$\bar{G}$	0.12	0.0027	0.17	0.58	9.02 (14.5; 5.56)	401 (171; 387)	6.55 (3.50; 12.2)	1.75 (1.41; 2.25)
	$\overline{ADM}$	0.12	0.0027	0.17	0.58	9.17 (15.1; 5.51)	409 (168; 387)	6.69 (3.46; 13.0)	1.76 (1.41; 2.29)
	$\overline{ADM}'$	0.12	0.0027	0.17	0.58	9.22 (15.4; 5.49)	409 (160; 385)	6.75 (3.40; 13.1)	1.77 (1.40; 2.32)
	$\overline{MDM}$	0.11	0.0027	0.14	0.53	10.4 (22.5; 4.59)	465 (115; 300)	9.68 (3.03; 29.0)	1.97 (1.35; 3.18)
	$\overline{MAD}$	0.12	0.0027	0.15	0.55	9.90 (19.7; 4.82)	444 (130; 325)	8.48 (3.13; 22.0)	1.89 (1.37; 2.85)
	$D7$	0.12	0.0027	0.17	0.58	9.22 (15.3; 5.50)	410 (165; 385)	6.73 (3.43; 13.1)	1.76 (1.40; 2.32)
	75	$\tilde{S}$	0.12	0.0027	0.18	0.60	8.67 (11.7; 6.43)	383 (235; 429)	5.77 (3.97; 8.39)
$\bar{S}$		0.12	0.0027	0.18	0.60	8.67 (11.7; 6.39)	383 (229; 429)	5.78 (3.96; 8.45)	1.68 (1.48; 1.94)
$\bar{S}_{25}$		0.12	0.0027	0.17	0.57	9.27 (15.8; 5.42)	412 (156; 380)	6.90 (3.38; 13.9)	1.78 (1.40; 2.36)
$\bar{S}_{20}$		0.12	0.0027	0.17	0.58	9.22 (15.0; 5.58)	408 (169; 394)	6.58 (3.46; 12.4)	1.75 (1.41; 2.26)
$\bar{R}$		0.12	0.0027	0.18	0.60	8.70 (12.0; 6.26)	384 (225; 425)	5.85 (3.90; 8.86)	1.69 (1.47; 1.98)
$\overline{IQR}$		0.12	0.0027	0.17	0.58	9.02 (14.6; 5.52)	402 (196; 386)	6.58 (3.48; 12.4)	1.75 (1.41; 2.26)
$\bar{G}$		0.12	0.0027	0.18	0.60	8.68 (11.7; 6.38)	384 (231; 428)	5.80 (3.95; 8.53)	1.68 (1.48; 1.95)
$\overline{ADM}$		0.12	0.0027	0.18	0.59	8.67 (11.9; 6.28)	386 (225; 427)	5.86 (3.95; 8.78)	1.69 (1.47; 1.97)
$\overline{ADM}'$		0.12	0.0027	0.18	0.60	8.70 (12.0; 6.28)	384 (219; 425)	5.86 (3.87; 8.84)	1.69 (1.46; 1.98)
$\overline{MDM}$		0.12	0.0027	0.17	0.58	9.23 (95.2; 33.2)	407 (138; 451)	6.58 (5.69; 39.3)	1.75 (2.13; 5.50)
$\overline{MAD}$		0.12	0.0027	0.17	0.58	8.96 (13.9; 5.73)	398 (185; 401)	6.35 (3.60; 11.2)	1.73 (1.43; 2.18)
$D7$		0.12	0.0027	0.18	0.60	8.69 (12.0; 6.28)	385 (223; 426)	5.84 (3.89; 8.81)	1.70 (1.46; 1.98)

TABLE 7. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Symmetric Variance Disturbances Are Present in Phase I for  $n = 5$ 

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.055	0.0043	0.016	0.11	23.0 (52.0; 7.68)	293 (475; 92.0)	195 (13.2; 427)	22.9 (3.22; 131)
	$\bar{S}$	0.041	0.0031	0.024	0.15	28.7 (55.8; 12.5)	359 (441; 158)	104 (11.5; 468)	9.45 (3.02; 30.8)
	$\bar{S}_{25}$	0.024	0.0024	0.040	0.20	52.7 (128; 19.4)	526 (164; 262)	84.4 (6.00; 478)	7.60 (2.19; 24.7)
	$\bar{S}_{20}$	0.024	0.0023	0.045	0.22	48.5 (103; 20.4)	493 (187; 275)	54.9 (6.50; 272)	5.97 (2.29; 16.6)
	$\bar{R}$	0.041	0.0032	0.023	0.15	28.4 (55.4; 12.2)	356 (445; 157)	110 (11.8; 483)	9.82 (3.05; 32.7)
	$\overline{\text{IQR}}$	0.024	0.0023	0.044	0.22	48.3 (103; 20.4)	493 (191; 276)	55.0 (6.56; 274)	5.99 (2.29; 16.2)
	$\bar{G}$	0.038	0.0029	0.026	0.16	30.2 (56.6; 14.0)	379 (429; 183)	86.0 (11.3; 388)	8.10 (2.99; 23.3)
	$\overline{\text{ADM}}$	0.035	0.0027	0.029	0.17	31.8 (58.9; 15.3)	393 (411; 200)	73.2 (10.8; 320)	7.28 (2.92; 19.3)
	$\overline{\text{ADM}}'$	0.024	0.0025	0.060	0.26	47.2 (86.6; 24.0)	450 (178; 330)	27.0 (6.41; 94.1)	4.31 (2.25; 8.80)
	$\overline{\text{MDM}}$	0.026	0.0023	0.041	0.21	46.6 (99.6; 19.3)	489 (213; 259)	63.5 (6.89; 323)	6.43 (2.35; 18.4)
	$\overline{\text{MAD}}$	0.033	0.0026	0.029	0.17	34.9 (69.6; 15.6)	423 (382; 206)	86.3 (9.68; 412)	8.02 (2.78; 23.8)
	$D7$	0.025	0.0024	0.055	0.25	44.1 (76.6; 23.9)	452 (230; 326)	27.8 (7.30; 90.8)	4.41 (2.40; 8.50)
	75	$\tilde{S}$	0.054	0.0042	0.012	0.11	20.4 (36.1; 10.3)	270 (467; 128)	163 (23.0; 478)
$\bar{S}$		0.040	0.0030	0.022	0.16	26.7 (41.7; 16.0)	353 (482; 210)	68.3 (17.3; 217)	7.86 (3.67; 14.8)
$\bar{S}_{25}$		0.024	0.0023	0.053	0.25	46.4 (83.1; 24.7)	473 (219; 339)	29.4 (7.04; 97.3)	4.55 (2.38; 9.01)
$\bar{S}_{20}$		0.025	0.0023	0.054	0.25	43.3 (27.4; 25.6)	459 (276; 351)	25.1 (8.08; 68.7)	4.25 (2.52; 7.49)
$\bar{R}$		0.041	0.0030	0.022	0.16	26.2 (41.2; 15.7)	347 (478; 205)	70.5 (17.5; 226)	7.30 (3.69; 15.2)
$\overline{\text{IQR}}$		0.025	0.0023	0.054	0.25	43.2 (70.1; 25.7)	459 (276; 351)	25.0 (8.07; 96.0)	4.25 (2.52; 7.48)
$\bar{G}$		0.037	0.0028	0.026	0.17	28.1 (42.7; 17.6)	371 (477; 233)	55.2 (16.2; 161)	6.41 (3.53; 12.2)
$\overline{\text{ADM}}$		0.035	0.0027	0.029	0.18	29.7 (44.5; 18.9)	387 (470; 253)	46.9 (15.1; 128)	5.89 (3.41; 10.6)
$\overline{\text{ADM}}'$		0.023	0.0023	0.065	0.28	44.7 (65.9; 29.5)	445 (269; 403)	18.2 (8.00; 39.3)	3.69 (2.52; 5.56)
$\overline{\text{MDM}}$		0.026	0.0023	0.049	0.24	41.5 (67.8; 24.3)	458 (299; 332)	28.4 (8.57; 80.6)	4.51 (2.59; 8.12)
$\overline{\text{MAD}}$		0.033	0.0025	0.031	0.19	32.2 (50.7; 20.0)	413 (462; 264)	45.8 (13.0; 135)	5.78 (3.18; 10.9)
$D7$		0.024	0.0022	0.059	0.27	42.7 (61.1; 29.2)	454 (321; 399)	19.5 (9.05; 40.3)	3.82 (2.66; 5.61)



TABLE 8. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Symmetric Variance Disturbances Are Present in Phase I for  $n = 9$

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.40	0.011	0.021	0.21	2.91 (6.13; 1.41)	147 (432; 39.2)	156 (10.2; 417)	8.78 (2.10; 36.4)
	$\bar{S}$	0.32	0.0068	0.033	0.27	3.55 (6.77; 1.81)	200 (463; 55.6)	77.4 (8.73; 384)	4.61 (1.98; 12.3)
	$\bar{S}_{25}$	0.16	0.0026	0.089	0.43	7.70 (17.6; 3.31)	449 (242; 197)	20.9 (3.93; 83.7)	2.58 (1.47; 5.03)
	$\bar{S}_{20}$	0.15	0.0025	0.10	0.46	7.92 (16.7; 3.63)	456 (233; 207)	15.8 (3.88; 54.6)	2.35 (1.47; 4.18)
	$\bar{R}$	0.35	0.0084	0.025	0.23	3.24 (6.52; 1.60)	175 (461; 41.8)	118 (9.69; 494)	6.19 (2.05; 20.4)
	$\overline{IQR}$	0.15	0.0025	0.10	0.46	7.79 (16.5; 3.62)	454 (240; 205)	15.9 (3.93; 55.3)	2.35 (1.47; 4.19)
	$\bar{G}$	0.27	0.0050	0.045	0.32	4.08 (7.38; 2.20)	246 (480; 84.8)	43.2 (7.69; 181)	3.52 (1.87; 7.45)
	$\overline{ADM}$	0.24	0.0042	0.054	0.35	4.56 (8.11; 2.52)	284 (490; 110)	31.2 (7.03; 114)	3.09 (1.82; 5.85)
	$\overline{ADM}'$	0.16	0.0027	0.12	0.49	7.10 (13.2; 3.72)	400 (236; 217)	11.5 (3.95; 31.7)	2.12 (1.48; 3.32)
	$\overline{MDM}$	0.15	0.0025	0.096	0.45	7.73 (16.5; 3.52)	452 (249; 197)	17.0 (3.96; 59.8)	2.41 (1.48; 4.36)
	$\overline{MAD}$	0.18	0.0029	0.076	0.41	6.21 (12.3; 3.07)	392 (392; 156)	22.0 (4.92; 80.3)	2.66 (1.59; 4.91)
	$D7$	0.15	0.0025	0.11	0.49	7.02 (12.0; 4.03)	415 (294; 247)	10.8 (4.32; 25.9)	2.09 (1.53; 3.05)
	75	$\tilde{S}$	0.41	0.010	0.016	0.20	2.61 (4.23; 1.65)	121 (262; 44.7)	123 (18.4; 414)
$\bar{S}$		0.32	0.0063	0.030	0.28	3.30 (5.00; 2.15)	180 (336; 82.0)	49.4 (13.1; 157)	3.85 (3.31; 7.00)
$\bar{S}_{25}$		0.16	0.0025	0.11	0.48	6.72 (11.4; 3.95)	414 (331; 241)	11.7 (4.62; 27.9)	2.15 (1.56; 3.12)
$\bar{S}_{20}$		0.15	0.0024	0.12	0.50	7.06 (11.3; 4.36)	426 (320; 279)	9.90 (4.56; 20.9)	2.04 (1.56; 2.79)
$\bar{R}$		0.36	0.0079	0.022	0.24	2.96 (4.60; 1.90)	149 (301; 62.6)	77.3 (15.6; 274)	4.71 (2.46; 9.68)
$\overline{IQR}$		0.15	0.0025	0.12	0.50	6.90 (11.0; 4.27)	417 (324; 271)	10.0 (4.59; 21.0)	2.04 (1.56; 2.80)
$\bar{G}$		0.27	0.0048	0.044	0.34	3.83 (5.59; 2.59)	229 (383; 118)	29.3 (10.5; 74.7)	3.11 (2.12; 4.86)
$\overline{ADM}$		0.24	0.0041	0.055	0.37	4.25 (6.10; 2.93)	266 (417; 146)	22.1 (9.23; 50.4)	2.80 (2.02; 4.07)
$\overline{ADM}'$		0.16	0.0025	0.12	0.52	6.64 (9.82; 4.44)	401 (331; 287)	8.99 (4.75; 16.8)	1.97 (1.58; 2.54)
$\overline{MDM}$		0.15	0.0025	0.11	0.49	6.88 (8.66; 3.61)	422 (443; 207)	10.4 (6.00; 30.7)	2.07 (1.71; 3.28)
$\overline{MAD}$		0.19	0.0029	0.086	0.44	5.60 (95.2; 33.2)	368 (138; 451)	13.9 (5.69; 39.3)	2.32 (2.13; 5.50)
$D7$		0.16	0.0025	0.12	0.51	6.58 (9.22; 4.65)	409 (370; 307)	8.85 (5.15; 15.2)	1.97 (1.66; 2.44)

TABLE 9. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Asymmetric Variance Disturbances Are Present in Phase I for  $n = 5$ 

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.16	0.020	0.016	0.074	16.1 (54.9; 1.46)	189 (444; 8.54)	211 (12.0; 34.0)	137 (3.08; 99.9)
	$\bar{S}$	0.063	0.0052	0.019	0.12	23.6 (57.8; 5.15)	286 (420; 56.2)	196 (11.1; 262)	43.3 (2.95; 447)
	$\bar{S}_{25}$	0.023	0.0024	0.044	0.21	55.1 (133; 20.5)	530 (140; 279)	74.8 (5.79; 421)	7.08 (2.13; 22.8)
	$\bar{S}_{20}$	0.025	0.0024	0.047	0.22	49.6 (106; 19.0)	488 (173; 249)	57.9 (6.30; 354)	6.88 (2.24; 20.2)
	$\bar{R}$	0.061	0.0050	0.019	0.12	23.8 (52.3; 5.51)	291 (420; 61.2)	195 (11.1; 284)	39.5 (2.95; 389)
	$\overline{\text{IQR}}$	0.025	0.0024	0.047	0.22	49.5 (107; 19.1)	490 (176; 254)	58.3 (6.36; 339)	6.72 (2.24; 19.6)
	$\bar{G}$	0.053	0.0042	0.021	0.13	25.8 (58.2; 6.94)	315 (407; 80.3)	168 (10.8; 375)	25.6 (2.91; 195)
	$\overline{\text{ADM}}$	0.047	0.0037	0.023	0.14	27.6 (60.0; 8.23)	334 (403; 98.5)	147 (10.5; 458)	18.8 (2.88; 113)
	$\overline{\text{ADM}}'$	0.022	0.0025	0.069	0.28	51.1 (90.7; 27.5)	448 (153; 378)	21.0 (5.94; 63.3)	3.86 (2.19; 7.12)
	$\overline{\text{MDM}}$	0.024	0.0023	0.045	0.22	48.8 (104; 20.5)	492 (181; 275)	56.0 (6.48; 279)	6.05 (2.25; 16.8)
	$\overline{\text{MAD}}$	0.046	0.0037	0.022	0.13	29.4 (68.9; 8.06)	352 (405; 96.4)	176 (9.89; 453)	23.9 (2.79; 166)
	$D7$	0.023	0.0024	0.062	0.27	47.5 (81.3; 26.5)	452 (197; 363)	22.6 (6.71; 65.1)	4.00 (2.32; 7.24)
	75	$\tilde{S}$	0.16	0.017	0.0079	0.038	10.5 (32.5; 1.91)	130 (432; 13.9)	263 (31.1; 61.8)
$\bar{S}$		0.059	0.0046	0.013	0.11	20.2 (39.5; 7.93)	264 (482; 93.3)	186 (19.1; 427)	19.7 (3.85; 95.0)
$\bar{S}_{25}$		0.023	0.0023	0.057	0.26	48.5 (86.8; 25.9)	473 (198; 343)	26.7 (6.71; 83.8)	4.31 (2.31; 8.27)
$\bar{S}_{20}$		0.025	0.0023	0.056	0.26	44.0 (72.8; 24.0)	453 (256; 328)	26.2 (7.64; 84.5)	4.27 (2.47; 8.15)
$\bar{R}$		0.058	0.0045	0.013	0.11	20.4 (39.4; 8.24)	268 (478; 99.1)	177 (19.2; 449)	17.6 (3.87; 78.1)
$\overline{\text{IQR}}$		0.025	0.0023	0.056	0.26	44.0 (72.7; 24.2)	453 (256; 332)	25.7 (7.66; 81.3)	4.27 (2.47; 8.02)
$\bar{G}$		0.050	0.0038	0.016	0.13	22.7 (41.2; 10.1)	299 (480; 124)	135 (17.5; 478)	12.1 (3.70; 43.3)
$\overline{\text{ADM}}$		0.046	0.0035	0.019	0.14	24.5 (42.9; 11.7)	320 (477; 147)	107 (16.2; 433)	9.66 (3.54; 29.6)
$\overline{\text{ADM}}'$		0.021	0.0024	0.075	0.30	48.4 (70.0; 32.9)	433 (231; 442)	15.3 (7.42; 30.2)	3.39 (2.42; 4.82)
$\overline{\text{MDM}}$		0.025	0.0023	0.054	0.25	43.5 (71.6; 25.5)	460 (268; 349)	25.4 (8.00; 72.2)	4.25 (2.52; 7.53)
$\overline{\text{MAD}}$		0.044	0.0034	0.019	0.14	25.7 (47.0; 11.7)	335 (492; 147)	116 (15.1; 472)	10.2 (3.42; 33.0)
$D7$		0.022	0.0023	0.069	0.29	46.1 (64.8; 32.2)	447 (278; 433)	16.4 (8.21; 31.9)	3.52 (2.55; 4.97)

TABLE 10. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Asymmetric Variance Disturbances Are Present in Phase I for  $n = 9$

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.69	0.11	0.025	0.093	1.90 (5.92; 1.00)	68.9 (412; 1.63)	176 (10.9; 8.33)	122 (2.14; 44.3)
	$\bar{S}$	0.48	0.021	0.021	0.17	2.66 (6.68; 1.09)	127 (460; 9.89)	192 (8.81; 123)	33.7 (1.97; 383)
	$\bar{S}_{25}$	0.15	0.0026	0.099	0.45	8.20 (18.6; 3.52)	462 (208; 198)	17.7 (3.67; 66.8)	2.43 (1.44; 4.60)
	$\bar{S}_{20}$	0.14	0.0025	0.11	0.47	8.28 (17.7; 3.63)	460 (204; 217)	14.9 (3.69; 51.7)	2.29 (1.44; 4.06)
	$\bar{R}$	0.50	0.024	0.017	0.15	2.52 (6.45; 1.07)	116 (453; 8.73)	214 (9.72; 107)	43.3 (2.06; 469)
	$\overline{IQR}$	0.14	0.0025	0.11	0.47	8.18 (17.4; 3.68)	459 (211; 268)	15.0 (3.71; 51.7)	2.29 (1.45; 4.05)
	$\bar{G}$	0.37	0.010	0.030	0.24	3.31 (7.26; 1.37)	179 (478; 26.5)	126 (7.77; 390)	8.78 (1.89; 41.9)
	$\overline{ADM}$	0.31	0.0070	0.038	0.28	3.81 (7.95; 1.63)	220 (491; 42.4)	87.5 (7.10; 490)	5.42 (1.82; 19.6)
	$\overline{ADM}'$	0.14	0.0026	0.14	0.53	7.95 (14.4; 4.34)	413 (190; 277)	9.00 (3.67; 21.6)	1.95 (1.43; 2.83)
	$\overline{MDM}$	0.14	0.0025	0.10	0.47	8.16 (17.2; 3.68)	461 (220; 216)	15.3 (3.77; 52.3)	2.31 (1.45; 4.08)
	$\overline{MAD}$	0.26	0.0052	0.049	0.32	4.79 (10.8; 1.82)	291 (479; 56.1)	74.2 (5.68; 517)	4.73 (1.67; 16.2)
	$D7$	0.14	0.0025	0.13	0.52	7.69 (12.9; 4.46)	426 (249; 291)	9.07 (4.02; 20.3)	1.96 (1.48; 2.74)
	75	$\tilde{S}$	0.77	0.092	0.012	0.044	1.42 (2.95; 1.00)	32.1 (150; 2.59)	205 (46.5; 18.5)
$\bar{S}$		0.49	0.017	0.012	0.15	2.27 (4.26; 1.25)	94.3 (269; 19.3)	204 (18.3; 272)	14.3 (2.63; 71.1)
$\bar{S}_{25}$		0.15	0.0025	0.12	0.50	7.18 (12.1; 4.21)	424 (288; 266)	10.4 (4.31; 23.8)	2.06 (1.52; 2.92)
$\bar{S}_{20}$		0.14	0.0024	0.13	0.52	7.36 (12.0; 4.47)	429 (292; 293)	9.33 (4.33; 19.8)	1.99 (1.52; 2.72)
$\bar{R}$		0.52	0.019	0.0093	0.13	2.11 (3.93; 1.20)	82.0 (241; 16.9)	233 (21.7; 233)	18.6 (2.86; 102)
$\overline{IQR}$		0.15	0.0025	0.12	0.51	7.20 (11.6; 4.40)	420 (294; 284)	9.28 (4.36; 19.9)	1.99 (1.53; 2.73)
$\bar{G}$		0.37	0.0086	0.024	0.24	2.97 (5.03; 1.68)	151 (342; 46.9)	92.8 (12.7; 395)	5.25 (2.29; 14.1)
$\overline{ADM}$		0.31	0.0064	0.034	0.29	3.41 (5.58; 1.99)	191 (385; 69.3)	54.7 (10.7; 226)	3.95 (2.13; 8.53)
$\overline{ADM}'$		0.14	0.0025	0.15	0.55	7.47 (10.8; 5.11)	408 (277; 347)	7.39 (4.32; 12.7)	1.83 (1.52; 2.27)
$\overline{MDM}$		0.15	0.0024	0.12	0.51	7.25 (11.8; 4.47)	428 (297; 288)	9.50 (4.38; 20.1)	2.00 (1.53; 2.72)
$\overline{MAD}$		0.26	0.0047	0.059	0.34	4.19 (7.21; 2.29)	256 (470; 92.0)	36.3 (7.70; 143)	3.29 (1.88; 6.71)
$D7$		0.14	0.0024	0.14	0.54	7.22 (10.0; 5.17)	416 (321; 351)	7.63 (4.70; 12.4)	1.86 (1.56; 2.26)

TABLE 11. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Localized Variance Disturbances Are Present in Phase I for  $n = 5$ 

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.10	0.0083	0.0038	0.035	12.1 (26.3; 4.82)	153 (362; 51.7)	370 (63.6; 243)	92.9 (7.10; 476)
	$\bar{S}$	0.051	0.0038	0.011	0.10	21.3 (36.8; 12.0)	286 (489; 152)	172 (27.0; 486)	13.1 (4.58; 34.6)
	$\bar{S}_{25}$	0.022	0.0024	0.047	0.22	57.2 (136; 21.8)	534 (129; 291)	66.1 (5.54; 363)	6.47 (2.09; 19.6)
	$\bar{S}_{20}$	0.051	0.0039	0.010	0.086	24.3 (54.9; 9.42)	317 (633; 115)	285 (18.4; 540)	27.4 (3.77; 133)
	$\bar{R}$	0.051	0.0038	0.011	0.10	21.4 (37.3; 11.8)	286 (492; 150)	176 (27.0; 499)	12.2 (4.59; 36.4)
	$\overline{\text{IQR}}$	0.051	0.0039	0.010	0.086	24.3 (55.0; 9.39)	316 (633; 115)	286 (18.4; 537)	28.1 (3.78; 134)
	$\bar{G}$	0.051	0.0038	0.011	0.10	21.4 (37.1; 11.9)	286 (450; 151)	174 (27.1; 491)	13.1 (4.62; 35.2)
	$\overline{\text{ADM}}$	0.051	0.0038	0.010	0.099	21.4 (37.1; 11.7)	287 (496; 149)	178 (26.4; 503)	13.5 (4.55; 37.2)
	$\overline{\text{ADM}}'$	0.020	0.0027	0.079	0.31	55.1 (97.2; 30.3)	433 (130; 415)	17.3 (5.51; 48.1)	3.53 (2.11; 6.16)
	$\overline{\text{MDM}}$	0.051	0.0039	0.010	0.086	24.3 (56.0; 9.25)	318 (635; 112)	288 (18.1; 532)	28.7 (3.75; 144)
	$\overline{\text{MAD}}$	0.051	0.0039	0.010	0.092	22.7 (46.0; 10.3)	301 (573; 128)	233 (21.7; 563)	18.9 (4.09; 72.3)
	$D7$	0.025	0.0023	0.053	0.25	43.6 (75.6; 24.2)	454 (240; 327)	27.9 (7.34; 85.2)	4.43 (2.44; 8.44)
	75	$\tilde{S}$	0.080	0.0064	0.0041	0.050	13.5 (22.9; 7.39)	175 (312; 87.4)	330 (73.0; 406)
$\bar{S}$		0.043	0.0032	0.017	0.14	24.1 (34.2; 16.7)	326 (450; 222)	77.6 (26.7; 186)	7.77 (4.57; 13.3)
$\bar{S}_{25}$		0.021	0.0024	0.065	0.28	51.7 (92.7; 27.8)	467 (165; 384)	22.5 (6.21; 67.9)	3.99 (2.21; 7.39)
$\bar{S}_{20}$		0.043	0.0032	0.016	0.13	25.3 (43.1; 14.1)	337 (520; 183)	112 (19.4; 377)	9.48 (3.85; 23.0)
$\bar{R}$		0.043	0.0032	0.017	0.14	24.0 (33.9; 16.5)	323 (448; 218)	78.1 (26.1; 188)	7.81 (4.53; 13.6)
$\overline{\text{IQR}}$		0.043	0.0032	0.016	0.13	0.043 (43.0; 14.1)	337 (519; 183)	112 (19.3; 375)	9.51 (3.85; 23.1)
$\bar{G}$		0.043	0.0032	0.017	0.14	23.9 (33.7; 16.6)	323 (447; 220)	77.6 (26.5; 186)	7.82 (4.55; 13.5)
$\overline{\text{ADM}}$		0.043	0.0032	0.017	0.14	23.9 (34.3; 16.4)	323 (449; 217)	78.9 (26.2; 195)	7.84 (4.52; 13.7)
$\overline{\text{ADM}}'$		0.020	0.0026	0.087	0.33	52.4 (74.6; 36.5)	405 (194; 474)	12.8 (6.72; 23.7)	3.13 (2.31; 4.30)
$\overline{\text{MDM}}$		0.043	0.0032	0.016	0.13	25.2 (43.3; 14.0)	336 (522; 180)	115 (19.3; 392)	9.69 (3.85; 24.1)
$\overline{\text{MAD}}$		0.043	0.0032	0.016	0.13	24.8 (38.7; 15.3)	333 (494; 139)	94.4 (22.1; 288)	8.60 (4.13; 18.0)
$D7$		0.023	0.0023	0.064	0.28	44.7 (63.1; 31.1)	452 (295; 423)	17.5 (8.55; 34.2)	3.63 (2.60; 5.19)

TABLE 12. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Localized Variance Disturbances Are Present in Phase I for  $n = 9$

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.66	0.033	0.0037	0.054	1.60 (2.60; 1.12)	43.1 (117; 12.0)	344 (93.7; 154)	49.3 (5.30; 263)
	$\bar{S}$	0.38	0.0088	0.015	0.20	2.75 (4.22; 1.86)	132 (462; 59.3)	111 (21.8; 346)	5.63 (2.82; 11.4)
	$\bar{S}_{25}$	0.13	0.0025	0.12	0.49	9.49 (21.5; 4.13)	481 (142; 255)	12.8 (3.24; 41.9)	2.17 (1.38; 3.74)
	$\bar{S}_{20}$	0.37	0.0089	0.015	0.18	3.09 (6.29; 1.61)	164 (469; 42.3)	201 (14.2; 590)	8.68 (2.40; 30.0)
	$\bar{R}$	0.38	0.0087	0.015	0.20	2.80 (4.41; 1.82)	136 (285; 57.5)	120 (20.9; 395)	5.86 (2.77; 12.7)
	$\overline{IQR}$	0.37	0.0089	0.014	0.18	3.09 (6.25; 1.61)	162 (467; 42.3)	202 (14.5; 591)	8.59 (2.42; 29.2)
	$\bar{G}$	0.38	0.0088	0.015	0.20	2.75 (4.23; 1.85)	133 (269; 59.4)	113 (21.7; 358)	5.68 (2.82; 11.7)
	$\overline{ADM}$	0.38	0.0087	0.015	0.20	2.79 (4.37; 1.84)	135 (276; 58.0)	117 (21.1; 379)	5.79 (2.79; 12.2)
	$\overline{ADM}'$	0.12	0.0028	0.17	0.57	9.21 (15.9; 5.33)	405 (147; 369)	6.86 (3.31; 14.0)	1.78 (1.39; 2.37)
	$\overline{MDM}$	0.37	0.0089	0.015	0.18	3.10 (6.27; 1.61)	164 (475; 42.2)	203 (14.4; 587)	8.78 (2.39; 30.6)
	$\overline{MAD}$	0.37	0.0088	0.015	0.18	2.97 (5.57; 1.66)	154 (406; 56.0)	175 (16.0; 567)	7.54 (2.50; 22.7)
	$D7$	0.16	0.0026	0.11	0.49	6.88 (11.7; 3.99)	412 (302; 243)	11.1 (4.42; 26.3)	2.11 (1.53; 3.09)
	75	$\tilde{S}$	0.58	0.021	0.0039	0.088	1.79 (2.51; 1.33)	55.0 (110; 24.2)	321 (85.3; 346)
$\bar{S}$		0.32	0.0063	0.027	0.27	3.19 (4.22; 2.43)	169 (264; 104)	44.6 (18.7; 97.0)	3.77 (2.66; 5.45)
$\bar{S}_{25}$		0.13	0.0025	0.15	0.55	8.48 (14.4; 4.97)	427 (195; 339)	7.82 (3.65; 16.3)	1.86 (1.44; 2.51)
$\bar{S}_{20}$		0.31	0.0062	0.026	0.26	3.36 (5.30; 2.17)	186 (371; 81.6)	61.7 (13.7; 205)	4.22 (2.36; 7.92)
$\bar{R}$		0.32	0.0063	0.027	0.27	3.20 (4.30; 2.39)	170 (273; 100)	46.3 (18.0; 106)	3.81 (2.62; 5.68)
$\overline{IQR}$		0.32	0.0064	0.026	0.26	3.31 (5.21; 2.14)	182 (361; 80.0)	61.6 (13.8; 199)	4.22 (2.37; 7.85)
$\bar{G}$		0.32	0.0063	0.027	0.27	3.19 (4.23; 2.43)	169 (266; 103)	45.0 (18.6; 97.9)	3.78 (2.66; 5.46)
$\overline{ADM}$		0.32	0.0063	0.026	0.27	3.19 (4.28; 2.40)	169 (270; 101)	46.1 (18.4; 104)	3.81 (2.64; 5.64)
$\overline{ADM}'$		0.12	0.0027	0.18	0.59	8.68 (12.2; 6.15)	384 (214; 421)	5.90 (3.83; 9.10)	1.69 (1.46; 1.99)
$\overline{MDM}$		0.31	0.0062	0.026	0.26	3.37 (5.28; 2.16)	187 (374; 82.0)	62.0 (13.8; 206)	4.22 (2.35; 7.96)
$\overline{MAD}$		0.32	0.0063	0.026	0.26	3.28 (4.91; 2.21)	179 (335; 85.8)	56.5 (14.9; 169)	4.09 (2.42; 7.15)
$D7$		0.15	0.0024	0.13	0.53	6.91 (9.61; 4.93)	414 (347; 332)	8.16 (4.92; 13.6)	1.91 (1.59; 2.44)

TABLE 13. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Diffuse Mean Disturbances Are Present in Phase I for  $n = 5$ 

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.055	0.0042	0.010	0.094	20.5 (38.8; 10.5)	271 (501; 131)	204 (23.7; 518)	15.8 (4.27; 49.0)
	$\bar{S}$	0.046	0.0035	0.015	0.12	24.1 (44.2; 12.4)	318 (519; 158)	139 (18.1; 477)	11.3 (3.76; 32.5)
	$\bar{S}_{25}$	0.027	0.0024	0.033	0.18	47.8 (117; 17.4)	509 (213; 230)	112 (6.82; 617)	9.29 (2.32; 33.7)
	$\bar{S}_{20}$	0.030	0.0025	0.031	0.18	40.3 (87.6; 15.9)	456 (291; 211)	97.3 (8.22; 509)	8.65 (2.53; 29.5)
	$\bar{R}$	0.047	0.0035	0.015	0.12	23.8 (43.9; 12.1)	316 (522; 155)	147 (18.4; 486)	11.8 (3.78; 34.1)
	$\overline{\text{IQR}}$	0.030	0.0025	0.031	0.18	40.3 (87.8; 16.0)	458 (293; 212)	96.7 (8.24; 504)	8.51 (2.53; 28.9)
	$\bar{G}$	0.044	0.0033	0.017	0.13	25.4 (45.7; 13.3)	336 (519; 172)	119 (16.8; 442)	9.98 (3.61; 26.7)
	$\overline{\text{ADM}}$	0.041	0.0031	0.019	0.14	26.7 (47.9; 14.0)	351 (514; 182)	107 (15.7; 393)	9.25 (3.51; 24.2)
	$\overline{\text{ADM}}'$	0.032	0.0028	0.041	0.20	38.2 (84.0; 15.5)	403 (202; 204)	67.7 (6.73; 337)	6.84 (2.32; 21.0)
	$\overline{\text{MDM}}$	0.030	0.0025	0.030	0.17	39.7 (87.0; 16.2)	459 (319; 213)	100 (8.52; 497)	8.69 (2.58; 28.0)
	$\overline{\text{MAD}}$	0.039	0.0029	0.019	0.14	29.5 (57.8; 13.7)	380 (531; 178)	128 (13.4; 510)	10.5 (3.26; 32.7)
	$D7$	0.031	0.0025	0.038	0.21	36.8 (69.0; 17.8)	422 (304; 237)	52.1 (8.45; 216)	5.98 (2.60; 14.8)
	75	$\tilde{S}$	0.054	0.0042	0.0091	0.097	19.2 (29.0; 12.7)	257 (393; 162)	156 (39.0; 383)
$\bar{S}$		0.046	0.0034	0.014	0.13	22.8 (33.7; 15.1)	307 (447; 197)	96.8 (27.2; 255)	8.79 (4.59; 16.8)
$\bar{S}_{25}$		0.026	0.0023	0.044	0.22	42.2 (76.3; 22.3)	468 (283; 302)	38.1 (3.21; 133)	5.14 (2.53; 10.6)
$\bar{S}_{20}$		0.030	0.0024	0.036	0.21	35.8 (59.9; 20.1)	434 (394; 274)	42.7 (10.6; 139)	5.48 (2.86; 11.1)
$\bar{R}$		0.047	0.0035	0.014	0.12	22.3 (33.3; 14.7)	301 (442; 192)	101 (27.5; 270)	9.00 (4.68; 17.4)
$\overline{\text{IQR}}$		0.030	0.0024	0.037	0.21	35.7 (59.9; 20.2)	434 (393; 275)	41.4 (10.5; 136)	5.45 (2.86; 11.0)
$\bar{G}$		0.044	0.0032	0.017	0.14	23.9 (35.0; 36.1)	322 (457; 211)	81.6 (24.5; 213)	7.96 (4.39; 14.5)
$\overline{\text{ADM}}$		0.041	0.0031	0.019	0.15	25.1 (36.5; 16.9)	338 (468; 223)	70.8 (22.4; 182)	7.36 (4.14; 13.2)
$\overline{\text{ADM}}'$		0.030	0.0025	0.043	0.22	36.1 (60.6; 20.0)	416 (321; 267)	36.3 (9.03; 119)	5.05 (2.69; 10.1)
$\overline{\text{MDM}}$		0.031	0.0024	0.035	0.20	35.3 (58.5; 20.4)	434 (411; 274)	43.9 (10.9; 140)	5.56 (2.92; 10.9)
$\overline{\text{MAD}}$		0.039	0.0029	0.021	0.15	27.3 (42.4; 17.1)	364 (504; 225)	71.1 (18.4; 208)	7.30 (3.78; 14.3)
$D7$		0.030	0.0024	0.040	0.22	35.4 (53.2; 22.6)	433 (410; 305)	32.1 (11.3; 82.0)	4.85 (2.97; 8.17)

TABLE 14. Marginal  $p$  and ARL and (in Parentheses) the Upper and Lower Conditional ARL Values when Diffuse Mean Disturbances Are Present in Phase I for  $n = 9$

$k$	$S_I$	$p$				ARL			
		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.41	0.010	0.014	0.18	2.62 (4.40; 1.67)	122 (282; 46.3)	149 (19.6; 461)	6.74 (2.70; 16.5)
	$\bar{S}$	0.36	0.0082	0.019	0.22	2.95 (4.97; 1.82)	150 (342; 56.7)	103 (15.0; 373)	5.40 (2.44; 12.4)
	$\bar{S}_{25}$	0.18	0.0031	0.069	0.38	6.51 (14.8; 2.75)	402 (343; 132)	32.9 (4.55; 154)	3.04 (1.55; 6.82)
	$\bar{S}_{20}$	0.18	0.0029	0.077	0.41	6.67 (14.3; 2.97)	411 (322; 150)	25.1 (4.52; 105)	2.76 (1.54; 5.64)
	$\bar{R}$	0.40	0.0098	0.014	0.19	2.72 (4.70; 1.66)	129 (310; 45.8)	152 (18.1; 486)	6.86 (2.61; 17.8)
	$\overline{IQR}$	0.18	0.0029	0.076	0.40	6.57 (14.2; 2.95)	409 (346; 147)	25.8 (4.62; 108)	2.79 (1.55; 5.72)
	$\bar{G}$	0.32	0.0065	0.027	0.26	3.33 (5.59; 2.04)	183 (391; 73.5)	66.5 (12.0; 246)	4.33 (2.23; 8.91)
	$\overline{ADM}$	0.29	0.0054	0.034	0.29	3.72 (6.24; 2.25)	218 (444; 90.0)	50.0 (10.3; 176)	3.81 (2.10; 7.44)
	$\overline{ADM}'$	0.23	0.0041	0.069	0.48	5.16 (11.3; 2.44)	345 (326; 104)	30.0 (4.56; 130)	2.97 (1.56; 6.19)
	$\overline{MDM}$	0.18	0.0029	0.075	0.40	6.60 (18.9; 3.01)	412 (360; 154)	25.1 (4.59; 104)	2.78 (1.56; 5.55)
	$\overline{MAD}$	0.23	0.0038	0.053	0.34	5.03 (9.93; 2.52)	322 (529; 110)	38.1 (6.23; 163)	3.30 (1.74; 7.02)
	$D7$	0.19	0.0031	0.081	0.42	5.69 (10.3; 3.02)	356 (387; 155)	18.3 (5.11; 59.2)	2.52 (1.61; 4.35)
75	$\tilde{S}$	0.41	0.010	0.012	0.19	2.49 (3.41; 2.86)	109 (189; 60.0)	112 (31.7; 284)	5.74 (3.32; 10.0)
	$\bar{S}$	0.36	0.0080	0.019	0.23	2.82 (3.89; 2.07)	137 (234; 74.8)	72.8 (22.4; 188)	4.64 (2.86; 7.68)
	$\bar{S}_{25}$	0.19	0.0029	0.081	0.43	5.75 (9.78; 3.37)	372 (428; 185)	16.4 (5.51; 44.2)	2.44 (1.66; 3.80)
	$\bar{S}_{20}$	0.18	0.0028	0.089	0.45	5.92 (9.63; 3.61)	384 (423; 205)	14.0 (5.48; 34.2)	2.31 (1.66; 3.41)
	$\bar{R}$	0.40	0.0097	0.013	0.19	2.55 (3.57; 1.87)	114 (203; 60.0)	108 (28.6; 290)	5.62 (3.17; 10.0)
	$\overline{IQR}$	0.18	0.0029	0.087	0.45	5.77 (9.32; 3.52)	373 (425; 197)	14.3 (5.58; 34.7)	2.33 (1.67; 3.44)
	$\bar{G}$	0.32	0.0064	0.027	0.27	3.18 (4.40; 2.33)	169 (281; 95.3)	47.4 (17.1; 114)	3.84 (2.57; 5.91)
	$\overline{ADM}$	0.29	0.0054	0.034	0.30	3.51 (4.83; 2.56)	199 (323; 113)	35.7 (14.0; 82.0)	3.41 (2.37; 5.03)
	$\overline{ADM}'$	0.22	0.0037	0.070	0.40	4.80 (7.98; 2.94)	301 (432; 147)	18.9 (6.24; 50.0)	2.59 (1.73; 4.03)
	$\overline{MDM}$	0.18	0.0028	0.087	0.45	5.87 (9.50; 3.61)	383 (430; 208)	14.2 (5.55; 34.0)	2.32 (1.67; 3.40)
	$\overline{MAD}$	0.23	0.0038	0.058	0.38	4.57 (7.04; 2.97)	292 (467; 149)	22.2 (8.05; 55.9)	2.78 (1.92; 4.24)
	$D7$	0.20	0.0030	0.085	0.44	5.30 (7.77; 3.59)	346 (439; 204)	13.7 (6.43; 28.4)	2.30 (1.76; 3.16)

TABLE 15. Measurements of Pitch Diameter of Threads on Aircraft Fittings

Sample	Observations					$S/c_4(5)$	Sample	Observations					$S/c_4(5)$
1	36	35	34	33	32	1.682	11	34	38	35	34	38	2.180
2	31	31	34	32	30	1.613	12	36	38	39	39	40	1.613
3	30	30	32	30	32	1.165	13	36	40	35	26	33	5.477
4	32	33	33	32	35	1.303	14	36	35	37	34	33	1.682
5	32	34	37	37	35	2.257	15	30	37	33	34	35	2.754
6	32	32	31	33	33	0.890	16	28	31	33	33	33	2.331
7	33	33	36	32	31	1.990	17	33	30	34	33	35	1.990
8	23	33	36	35	36	5.856	18	27	28	29	27	30	1.387
9	43	36	35	24	31	7.424	19	35	36	29	27	32	4.079
10	36	35	36	41	41	3.138	20	33	35	35	39	36	2.331

most quickly when diffuse or localized disturbances occur. Because of this risk, we do not recommend using  $\tilde{S}$ .

- The charts based on the estimators  $\overline{S}_{20}$ ,  $\overline{IQR}$ , and  $\overline{MDM}$  perform relatively well in response to diffuse disturbances but not very well when there are no contaminations.
- The charts based on estimators  $\overline{G}$  and  $\overline{ADM}$  are efficient under normality and are more efficient than the traditional charts based on  $\tilde{S}$ ,  $\overline{S}$ , and  $\overline{R}$  when diffuse outliers are present.
- The charts based on the estimators  $\overline{ADM}'$  and  $D7$  perform equally well as the traditional charts in the uncontaminated case and substantially better than any of the other charts in contaminated situations. When mean diffuse disturbances are likely to occur in Phase I, we recommend using  $D7$  because the control chart based on this estimator is more robust against such (extreme) disturbances. When localized disturbances, i.e., disturbances that affect an entire sample, are likely to occur, we recommend using  $\overline{ADM}'$ . Advantages of the latter estimator are the ease of obtaining estimates and its intuitiveness: extreme samples and, hence, the root cause of any disturbances can be readily identified.

### Real-Data Example

In this section, we demonstrate the implementation of the control charts created above. Our dataset was supplied by Grant and Leavenworth (1988, p. 9). The operation concerns thread grinding a fitting for an aircraft hydraulic system. Table 15 shows the

pitch diameters of the threads for 20 randomly chosen samples. Each sample consists of 5 observations.

The control-charting process starts with estimating the in-control standard deviation  $\sigma$  (Phase I). We construct control charts based on the different Phase I estimators proposed. The estimates derived from these estimators are shown in Table 16. Based on the Phase I estimates, the Phase II control limits are determined. For example, the estimate of  $\sigma$  based on  $\tilde{S}$  is equal to 2.972, and Table 3 shows that the respective factors for the upper and lower control limits are 2.352 and 0.171. Consequently the Phase II control limits are 6.990 and 0.508. Figure 5 compares the Phase II control limits for the proposed estimators.

TABLE 16. Control-Chart Limits for Pitch Diameters

$S_I$	$\hat{\sigma}$	$\widehat{UCL}$	$\widehat{LCL}$
$\tilde{S}$	2.972	6.990	0.508
$\overline{S}$	2.657	6.263	0.112
$\overline{S}_{25}$	2.193	5.930	0.366
$\overline{S}_{20}$	2.456	6.238	0.415
$\overline{R}$	2.666	6.302	0.456
$\overline{IQR}$	2.424	6.159	0.410
$\overline{G}$	2.623	6.188	0.449
$\overline{ADM}$	2.594	6.137	0.444
$\overline{ADM}'$	2.041	4.849	0.349
$\overline{MDM}$	2.256	5.762	0.381
$\overline{MAD}$	2.408	5.892	0.409
$D7$	2.067	4.911	0.353



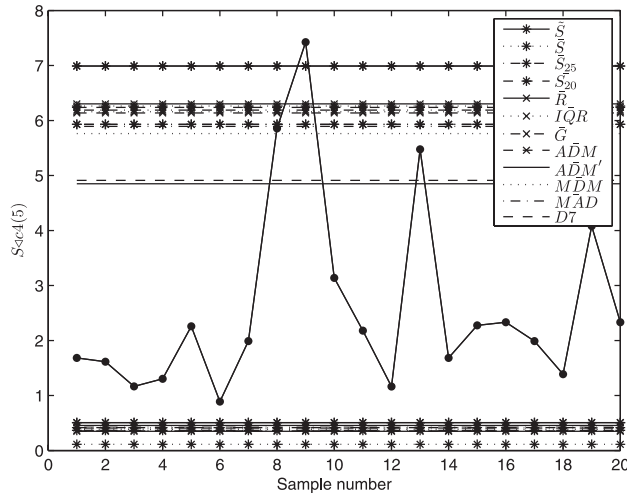


FIGURE 5. Control-Chart Limits for Pitch Diameters.

In the case of  $\overline{ADM}'$ , we apply a simple subgroup-screening method. The factors for the Phase I control limits are 2.089 and 0 for  $n = 5$ . We first determine the  $\overline{ADM}$  from the 20 subsamples, which generates 2.594. The Phase I control limits are 5.419 and 0. Then we determine the standard deviation  $S/c_4(5)$  of each subsample and delete subsamples for which the standard deviation falls outside the initial control limits. For the example discussed here, the standard deviations of subsamples 8, 9, and 13 fall outside the control limits. The same procedure is repeated in the second iteration: new values for the in-control  $\sigma$  (2.041) and the Phase I control limits (4.263 and 0) are generated from the remaining subsamples and any subsample for which the standard deviation falls outside the control limits is deleted. In the second iteration, it appears no longer necessary to delete further subsamples.

The estimator  $\tilde{S}$  gives the highest  $\widehat{UCL}$  and  $\widehat{LCL}$ . The estimators that give the lowest  $\widehat{UCL}$  and  $\widehat{LCL}$  are  $D7$  and  $\overline{ADM}'$ . Note, however, that the question of which estimator gives the best estimate can not be resolved from such a limited sample.

### Concluding Remarks

In this article, we have compared 12 different estimators for designing the control chart for the standard deviation and investigated their performance in Phase II. The added value of incorporating a simple screening procedure into an estimation method turned out to be substantial. This method performed

better than estimators that remove samples ( $\overline{S}_{25}$ ) or observations ( $\overline{S}_{20}$  or  $\overline{IQR}$ ) beforehand. The disadvantage of removing samples and/or observations beforehand is that too much information is lost in uncontaminated situations while, at the same time, the resulting estimates are biased in contaminated situations. The estimator  $\overline{ADM}'$  uses a great deal of information, deleting only extreme subgroups so that the final estimate is not affected substantially. Moreover,  $\overline{ADM}'$  is intuitive and easy to implement. We recommend using  $\overline{ADM}'$  when the dataset is likely to be contaminated by localized disturbances, i.e., disturbances that affect an entire sample. On the other hand, we prefer  $D7$  when the dataset is likely to be contaminated by mean diffuse disturbances because  $D7$  is more robust against such disturbances. There is no single best control-chart method that would cover every process and every company. ASTM 15D (1976, p.143) says it best: “The final justification of a control chart criterion is its proven ability to detect assignable causes economically under practical conditions.”

### Appendix A

The literature proposes several estimators for the standard deviation of a normal distribution, including estimators based on Gini’s mean differences, Downton’s linear function of order statistics (Downton (1966)), and the probability-weighted moments estimator (Muhammad et al. (1993)).

Let  $X_{i(1)}, X_{i(2)}, \dots, X_{i(n)}$  denote the order statistics of sample  $i$ . According to David (1968), the sample statistic  $G_i$  can also be written as a function of order statistics,

$$G_i = 2/(n(n - 1)) \sum_{j=1}^n (2j - n - 1)X_{i(j)}. \quad (19)$$

Downton (1966) suggests as a possible unbiased estimator of  $\sigma$  the statistic

$$D_i = 1/\sqrt{\pi} \sum_{j=1}^n (2j - n - 1)X_{i(j)}/(n(n - 1)), \quad (19)$$

and Muhammad et al. (1993) proposes the so-called probability weighted-moments estimator of  $\sigma$ ,

$$S_{pw,i} = \sqrt{\pi}/n^2 \sum_{j=1}^n (2j - n - 1)X_{i(j)}. \quad (21)$$

It follows directly from (19), (20), and (21) that

$$G_i = 2/\sqrt{\pi}D_i = 2n/((n - 1)\sqrt{\pi})S_{pw,i}.$$

### Appendix B

One of our estimators of the sample standard deviation is based on the average absolute deviation from the median,  $ADM_i$ . As is true for  $G_i$ , we can write  $ADM_i$  as a function of order statistics,

$$nADM_i = \begin{cases} X_{i(n)} + \dots + X_{i((n+3)/2)} - X_{i((n-1)/2)} - \dots - X_{i(1)} & \text{if } n \text{ is odd} \\ X_{i(n)} + \dots + X_{i(n/2+1)} - X_{i(n/2)} - \dots - X_{i(1)} & \text{if } n \text{ is even.} \end{cases} \tag{22}$$

From Equations (19) and (22), we can easily derive the relationship between  $G_i$  and  $ADM_i$ ,

$$G_i = \begin{cases} 2ADM_i + \frac{2}{n(n-1)} \left( 2X_{i(2)} + 4X_{i(3)} + 6X_{i(4)} + \dots + (n-3)X_{i(\frac{n-1}{2})} - (n-3)X_{i(\frac{n+3}{2})} - \dots - 6X_{i(n-3)} - 4X_{i(n-2)} - 2X_{i(n-1)} \right) & \text{if } n \text{ is odd} \\ 2ADM_i + \frac{2}{n(n-1)} \left( 2X_{i(2)} + 4X_{i(3)} + 6X_{i(4)} + \dots + (n-2)X_{i(\frac{n}{2})} - (n-2)X_{i(\frac{n}{2}+1)} - \dots - 6X_{i(n-3)} - 4X_{i(n-2)} - 2X_{i(n-1)} \right) & \text{if } n \text{ is even.} \end{cases}$$

From David (1981, p. 192), it follows that the estimator based on the average absolute deviation from the median is less efficient than the estimator based on Gini’s mean differences.

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