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LETTER TO THE EDITOR

On guaranteed in-control performance for the Shewhart X and $\bar{X}$ control charts

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ABSTRACT
Recently, two methods have been published in this journal to determine adjusted control limits for the Shewhart control chart in order to guarantee a pre-specified in-control performance. One is based on the bootstrap approach (Saleh et al. (2015)), and the other is an analytical approach (Goedhart, Schoonhoven, and Does (2017)). Although both methods lead to the desired control chart performance, they are still difficult to implement by the practitioner. The bootstrap is rather computationally intensive, while the analytical approach consists of multiple integrals and derivatives. In this letter to the editor we simplify the analytical expressions provided in Goedhart, Schoonhoven, and Does (2017) by using the theory on tolerance intervals for individual observations as given in Krishnamoorthy and Mathew (2009).

KEYWORDS
control limits; tolerance intervals; bootstrap; statistical process monitoring

1. Adjusted control limits

Goedhart, Schoonhoven, and Does (2017) have derived adjusted control limits for the Shewhart X and $\bar{X}$ control charts to guarantee a specified conditional in-control performance. Their proposed analytical adjustments are quite accurate and are an improvement to other computationally intensive methods such as the recently proposed bootstrap method (e.g., Saleh et al. 2015). However, the analytical expressions can still be difficult to implement. To this end, a simpler formula to achieve the same performance has been derived based on the tolerance interval theory in Krishnamoorthy and Mathew (2009). In Goedhart, Schoonhoven, and Does (2017), this theory has been used to derive a simple formula for the individual Shewhart X chart. Although tolerance intervals are intended for use on individual observations, their theory can be extended to also be applicable to the $\bar{X}$ chart. In this article, we provide a simple formula to guarantee a similar performance to that in Goedhart, Schoonhoven, and Does (2017).

2. Tolerance approximation to construct estimated lower control limit (LCL) and upper control limit (UCL)

As in Goedhart, Schoonhoven, and Does (2017), we define X_{ij} as the jth observation in sample i ($i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$). We assume that X_{ij} are independently $N(\mu, \sigma)$ distributed. The process parameters μ and σ in Phase I are estimated from m samples of size n, with their estimates denoted as $\hat{\mu}$ and $\hat{\sigma}$, respectively. For Phase II, $\bar{X}$ denotes the sample mean of a monitoring sample ($i = m+1,\ m+2, \ldots$) and k denotes the factor for the control limits required to obtain the desired conditional performance.

Note that the probability of no signal for the $\bar{X}$ chart (i.e., $\bar{X}$ is between $\widehat{LCL}$ and $\widehat{UCL}$) for normal data is equal to

$$P\left(\bar{X} < \hat{\mu} + k\frac{\hat{\sigma}}{\sqrt{n}}\right) - P\left(\bar{X} < \hat{\mu} - k\frac{\hat{\sigma}}{\sqrt{n}}\right) = \Phi\left(\frac{\hat{\mu}-\mu}{\sigma/\sqrt{n}} + k\frac{\hat{\sigma}}{\sigma}\right) - \Phi\left(\frac{\hat{\mu}-\mu}{\sigma/\sqrt{n}} - k\frac{\hat{\sigma}}{\sigma}\right) \quad [1]$$

where $\Phi(x)$ denotes the standard normal cumulative distribution function (CDF). Consider a general unbiased estimator for location that follows, either exactly or approximately, a normal distribution when the data are normally distributed, such as the grand sample average or the grand sample median. In that case, we have $\hat{\mu} \sim N(\mu, \sigma_{\hat{\mu}})$. This means that $Z = \frac{\hat{\mu}-\mu}{\sigma/\sqrt{n}} \sim N(0, \frac{\sigma_{\hat{\mu}}}{\sigma/\sqrt{n}})$. Also, consider an estimator $\hat{\sigma}$ such that $W = \frac{\hat{\sigma}}{\sigma} \sim \frac{a\chi_b}{\sqrt{b}}$, either exactly or approximately, with a and b some constants whose values depend on the estimator $\hat{\sigma}$. Then we can rewrite Eq. [1] as

$$\Phi(Z + kW) - \Phi(Z - kW). \quad [2]$$

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The goal is to find the value of k that provides an in-control conditional average run length (ARL) of at least $1/\alpha_{tol}$ with probability $1-p$, where α_{tol} is typically a small value such as 0.0027. Note that this is equivalent to a false alarm rate (FAR) of at most α_{tol} with probability $1-p$, which is in turn equivalent to having a probability of no signal of at least $1-\alpha_{tol}$ with probability $1-p$. Mathematically, this can be written as

$$P_{Z,W}\left(\Phi\left(Z+kW\right)-\Phi\left(Z-kW\right)\geq 1-\alpha_{tol}\right)=1-p. \quad [3]$$

Result 1.2.1 in Krishnamoorthy and Mathew (2009) states that for $Z\sim N(0,\sqrt{c})$ independently of $Q\sim\frac{\chi^2_\nu}{\nu}$, where χ^2_ν denotes a Chi-square random variable with degrees of freedom (df) ν, an approximate solution for the value k_2 such that

$$P_{Z,Q}(\Phi(Z+k_2\sqrt{Q})-\Phi(Z-k_2\sqrt{Q})\geq 1-\alpha_{tol})=1-p \quad [4]$$

is given by

$$k_2=\left(\frac{\nu\chi^2_{1;1-\alpha_{tol}}(c)}{\chi^2_{\nu;p}}\right)^{1/2} \quad [5]$$

where $\chi^2_{\nu;\gamma}(\delta)$ denotes the γ quantile of a noncentral Chi-square distribution with df ν and noncentrality parameter δ, and where $\chi^2_{\nu;\gamma}$ denotes the γ quantile of a Chi-square distribution with df ν.

To this end, we can write Eq. [3] as

$$P_{Z,W}\left(\Phi\left(Z+ka\frac{W}{a}\right)-\Phi\left(Z-ka\frac{W}{a}\right)\geq 1-\alpha_{tol}\right)=1-p \quad [6]$$

Next, since $\frac{W}{a}\sim\frac{\chi_b}{\sqrt{b}}$, we use the result derived by Krishnamoorthy and Mathew (2009) to find the desired value of k, namely

$$k=\left(\frac{b\chi^2_{1;1-\alpha_{tol}}(c)}{a^2\chi^2_{b;p}}\right)^{1/2} \quad [7]$$

where $\sqrt{c}=\frac{\sigma_{\hat{\mu}}}{\sigma/\sqrt{n}}$.

As an example, consider $\hat{\mu}=\bar{\bar{X}}$ and $\hat{\sigma}=S_p$. This means that we have $\sqrt{c}=\frac{\sigma_{\hat{\mu}}}{\sigma/\sqrt{n}}=\frac{\sigma/\sqrt{mn}}{\sigma/\sqrt{n}}=1/\sqrt{m}$, $a=1$, and $b=m(n-1)$. We then find the desired control charting constant to be

$$k_{\bar{\bar{X}},S_p}=\left(\frac{m(n-1)\chi^2_{1;1-\alpha_{tol}}(1/m)}{\chi^2_{m(n-1);p}}\right)^{1/2}. \quad [8]$$

Note that the required quantiles, and consequently this formula, can easily be calculated in common statistical software programs such as R or Matlab. Note also that similar expressions for k can easily be obtained for different estimators. The only restrictions are that the estimator for location (approximately) follows a normal distribution and that the estimator for dispersion (approximately) follows a scaled chi distribution. This holds true for many other commonly used estimators, such as the grand sample median for location and the average sample standard deviation, the average sample range, and the grand sample standard deviation for dispersion. For more information on this, we refer to Goedhart, Schoonhoven, and Does (2017).

3. Performance of the adjusted limits

In Tables 1 and 2 we illustrate the adjusted control limit factors and their resulting exceedance probabilities (the fraction of the control charts with a conditional in-control ARL below the prespecified threshold ARL_0) for two different sets of parameters. All exceedance probabilities are determined based on 100,000 simulated Phase I samples. Note that the values should by design be close to p. As can be observed from the tables, the difference between the approximation in Goedhart, Schoonhoven, and Does (2017) and the newly provided approximation is negligible. We obtain similar performances for

Table 1. Adjusted control limit factors k and exceedance probabilities (EP) for Goedhart, Schoonhoven, and Does (2017) (subscript G) and for the new factors (subscript KM). Parameter values are $\alpha_{tol}=0.0027$, $p=0.1$, $n=5$, and various values of m indicated in the table.

m	k_G	EP_G	k_{KM}	EP_{KM}
25	3.3827	0.0969	3.3687	0.1060
50	3.2473	0.0966	3.2399	0.1042
100	3.1640	0.0962	3.1595	0.1030
150	3.1291	0.0985	3.1266	0.1030
200	3.1094	0.0974	3.1077	0.1009
300	3.0870	0.0986	3.0862	0.1006
500	3.0657	0.0995	3.0654	0.1007
1000	3.0454	0.0993	3.0453	0.1000

Table 2. Adjusted control limit factors k and exceedance probabilities (EP) for Goedhart, Schoonhoven, and Does (2017) (subscript G) and for the new factors (subscript KM). Parameter values are $\alpha_{tol}=0.01$, $p=0.05$, $n=5$, and various values of m indicated in the table.

m	k_G	EP_G	k_{KM}	EP_{KM}
25	2.9665	0.0607	2.9743	0.0568
50	2.8315	0.0575	2.8357	0.0547
100	2.7478	0.0547	2.7492	0.0535
150	2.7124	0.0539	2.7137	0.0526
200	2.6922	0.0543	2.6933	0.0525
300	2.6691	0.0537	2.6700	0.0519
500	2.6468	0.0505	2.6474	0.0489
1000	2.6251	0.0509	2.6255	0.0497

other parameter values and common standard deviation estimators.

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