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Research

EWMA Control Chart Limits for First- and Second-Order Autoregressive Processes[‡]

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Today's manufacturing environment has changed since the time when control chart methods were originally introduced. Sequentially observed data are much more common. Serial correlation can seriously affect the performance of the traditional control charts. In this article we derive explicit easy-to-use expressions of the variance of an EWMA statistic when the process observations are autoregressive of order 1 or 2. These variances can be used to modify the control limits of the corresponding EWMA control charts. The resulting control charts have the advantage that the data are plotted on the original scale making the charts easier to interpret for practitioners than charts based on residuals.

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INTRODUCTION

Today's manufacturing environment no longer resembles that in which control chart methods originally were introduced. Owing to the widespread use of automated sensors, data are often sampled sequentially in time and with a sampling rate that can be very high. Hence, the assumption of independence of successive observations is often violated. Serial correlation may seriously affect the performance of traditional control charts based on the assumption of independent and identically distributed observations. Thus, serial correlation should not be ignored.

This raises a more fundamental question of what is required for a process to be in a state of statistical control. According to ANSI/ISO/ASQC Standard A3534-2-1993, statistical control is defined as '[A] state in which the variations among the observed sampling results can be attributed to a system of chance causes that does not appear to change with time.' There is general agreement that this definition implies that when in a state of statistical control, the mean and standard deviation of the process remain constant over time. The definition is also sometimes interpreted to imply that consecutive observations are statistically

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independent. However, this latter condition is unnecessarily restrictive. A weakly stationary time series exhibits a probability distribution with the first two moments constant. Hence, we would consider it to be in statistical control. See also Alwan and Roberts[1] for a general discussion.

Besides the problem of autocorrelated observations, there is a wide range of other interesting research issues. Montgomery[2] describes that statistical process control (SPC) has been called one of the most important developments of the 20th century and its contribution to decrease the process variability is enormous. SPC is a very important and active research field. For the past couple of years, a number of papers have appeared on problems arising from typical operating processes[3–6], reviews and extensions of existing theories[7–11], and papers on the problem of correlated observations[12–15].

MacGregor[16] argued that some or all autocorrelations should be eliminated by feedback control. Although obviously desirable, that may in practice not always be economical. For example, if the dead-time is long, the relationship between a control action and a process reaction is relatively weak and when the inertia of the process is relatively large, feedback control may not be worth the trouble. Indeed, applying feedback control, if not done right, may increase rather than decrease the variability. Thus, it may be better to resign to the fact that a stationary but autocorrelated process constitutes a stable and predictable common cause system. However, stationary but autocorrelated processes may wander randomly around its long-term mean without necessarily being out of control. The function of a control chart will then be to monitor the process to see whether it remains stationary or otherwise to sound an alarm. Of course, in this context we do not want to imply that we believe that processes will remain stationary for extended periods of time; see Box[17]. Rather we assume only that the process behaves, for the time being, though if it is stationary and the function of the control chart is to alert us when this provisional assumption no longer appears to be tenable.

Alwan and Roberts[1] provided a comprehensive overview and discussion of modern process monitoring methods when processes follow a linear discrete time series model; see also Box and Luceño[18]. Lu and Reynolds[19] provide a more recent overview of control charts for processes with autocorrelated data. Essentially three approaches have been promoted in the past.

One approach is to use standard control charts with suitably modified control limits taking into account the autocorrelation; Zhang[20,21], Jiang *et al.*[22], and Apley and Lee[23], among others, adopted this approach. The practical appeal is that it is relatively simple to explain, the familiar 'good old methods' can be used avoiding retraining and fear of new methods, and it allows operators to retain their physical intuition about the process. This approach may, of course, lead to charts that are less sensitive to detecting small process changes. However, in many practical applications it is more important to detect real significant changes and avoid many false alarms caused by minor fluctuations in the data.

Another approach is to fit an appropriate time series model to the data and monitor the residuals with standard control charts, for example, individual charts, $\bar{X}$ charts, or EWMA charts. Berthouex *et al.*[24] is an early reference; see also Montgomery and Mastrangelo[25], Wardell *et al.*[26], and Koehler *et al.*[27]. Yet another approach is to use two charts, one for the residuals after fitting a time series model and another based on the one-step-ahead predictions; see Alwan and Roberts[1] and Alwan[28]. The latter two approaches are less intuitive, because the residuals can be difficult to interpret and are not presented in the original scale of the measurements.

In industrial practice, (temporarily) stationary processes can often be modeled as an autoregressive process of order 1 or 2; see Box *et al.*[29] (p. 98). Higher-order processes or processes better modeled with additional moving average terms may, of course, occur but are less common; see Alwan and Roberts[1]. Thus, a closed-form expression for the variance of the EWMA statistic under an AR(2) process, with the AR(1) process as a special case, will cover not all but a significant number of practical applications.

The EWMA control chart is frequently used without any reference to the underlying process. In other cases it is implied that the observations are independent identically distributed (IID) with a normal distribution. The EWMA control chart when used for control is based on the assumption that the process is IID. However, in practice EWMA control charts may, for a number of reasons, be used even if the process is not IID. One major reason is that operators may be familiar and comfortable with the use of the EWMA control chart. Thus, the rationale may be that it is better to use a slightly inappropriate method than none at all, especially if the inappropriate method can be modified to perform reasonably well.

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In this article, we present an explicit expression for the variance of the EWMA statistic for an AR(2) process. This expression is dependent on the monitoring time. By setting model parameters to zero, our expression will also apply to an AR(1) process or the IID case. Using our closed-form expression of the asymptotic variance (i.e. assuming that the monitoring time is large), we can relatively easily establish control limits for a Phase II application of an EWMA control chart. The use of this modified EWMA control chart will be illustrated below with the application to data from a ceramic furnace; see Bisgaard and Kulahci[30].

Other relevant literature includes Vasilopoulos and Stamboulis[31]. They calculated the exact variance for the $\bar{X}$ statistic assuming an AR(2) process. Based on this variance, appropriate control limits can be derived. Schmid[32] and VanBrackle and Reynolds[33] derived the asymptotic variance, and Wieringa[34] derived the exact variance for the EWMA statistic under the assumption of an AR(1) process.

In Zhang[20] the variance of the EWMA statistic is studied for an AR(p) process. Zhang expressed the variance as an infinite sum of autocorrelation coefficients of the AR(p) process. However, the expressions are not in a closed form as are those derived by Schmid[32], VanBrackle and Reynolds[33], and Wieringa[34].

Note that the control limits obtained with the use of our closed-form expression are asymptotically the same as those obtained from Zhang's expression[20]. The difference between these two expressions is the estimation method. A comparison of the different estimation methods based on the asymptotic relative efficiency of the estimators is made by Vermaat *et al.*[35]. They showed that the closed-form expression is statistically more efficient than Zhang's expression[20].

Zhang[20] also compared the average run lengths (ARLs) of his control chart with the residual chart and the individual chart for AR(2) processes (see Table 2 in Zhang[20]). From his study it appears that if the process is not nearly non-stationary then Zhang's control[20] chart has a better performance. However, if the process is nearly non-stationary then the residual chart performs better. Hence, it seems reasonable to assume that the performance of the control chart proposed in this article for stationary processes will be even better than that of Zhang[20].

In the following section we briefly introduce the relevant time series theory. Then follows a section where we derive an explicit asymptotic expression for the variance of the EMWA statistic for an AR(2) process and discuss the consequences. Details of the proof can be found in Appendix A, together with additional properties of this variance estimator in Appendix B. The modified EWMA control chart, using the asymptotic expression for the variance, is then applied to a real-life example. We end the article with concluding remarks.

A BRIEF SUMMARY OF TIME SERIES THEORY

We first provide a brief summary of relevant time series results. For a more complete description, see, e.g. Box *et al.*[29]. Let $\{z_t\}$, $t=1,2,\ldots,$ be a stationary time series process. Let μ be the mean and $\tilde{z}_t = z_t - \mu$, then the AR(2) process is defined as

$$\tilde{z}_t = \phi_1 \tilde{z}_{t-1} \quad \phi_2 \tilde{z}_{t-2} \quad a_t, \quad t=1,2,\ldots$$

where a_t is white noise, i.e. a_t is a sequence of uncorrelated random variables with mean zero and constant variance. Most time series texts provide expressions for the autocorrelation. However, we need an expression for the autocovariance. The first two autocovariances of the AR(2) process are

$$\gamma_0 = \frac{(\phi_2 - 1)\sigma_a^2}{(1 \quad \phi_2)(\phi_1^2 - (1-\phi_2)^2)} \tag{1}$$

and

$$\gamma_1 = \frac{-\phi_1 \sigma_a^2}{(1 \quad \phi_2)(\phi_1^2 - (1-\phi_2)^2)} \tag{2}$$

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The covariance function of the AR(2) process is

$$\gamma_k = \begin{cases} \dfrac{v_1^k(\gamma_1 - v_2\gamma_0)}{v_1 - v_2} - \dfrac{v_2^k(\gamma_1 - v_1\gamma_0)}{v_1 - v_2} & \text{if } v_1 \neq v_2 \\ v^{k-1}[k(\gamma_1 - v\gamma_0) \quad v\gamma_0] & \text{if } v_1 = v_2 = v \end{cases} \tag{3}$$

where v_1 and v_2 are the roots of the characteristic function $\pi^2 - \phi_1\pi - \phi_2 = 0$; see Fuller[36] (pp. 54–56). The roots v_1 and v_2 can be computed explicitly from the parameters ϕ_1 and ϕ_2 using the following expressions:

$$v_1 = \frac{\phi_1}{2} \quad \frac{\sqrt{(\phi_1^2 \quad 4\phi_2)}}{2} \tag{4}$$

$$v_2 = \frac{\phi_1}{2} - \frac{\sqrt{(\phi_1^2 \quad 4\phi_2)}}{2} \tag{5}$$

The AR(2) process is stationary if the roots v_1 and v_2 are within the unit circle, $|v_1| < 1, |v_2| < 1$. Note that the roots of the characteristic equation are complex if $\phi_1^2 \quad 4\phi_2 < 0$. The autocovariance function (3) can in that case be expressed as

$$\gamma_k = \frac{r^{k-1}[\gamma_1 \sin k\theta - \gamma_0 r \sin(k-1)\theta]}{\sin\theta}$$

where $v_1 = re^{i\theta}$ and $v_2 = re^{-i\theta}$.

CONTROL LIMITS FOR THE EWMA CONTROL CHART FOR AN AR(2) PROCESS

The EWMA statistic is defined as

$$\begin{aligned} W_{\tilde{z},t} &= \lambda\tilde{z}_t \quad (1-\lambda)W_{\tilde{z},t-1} \\ &= \lambda\sum_{i=0}^{t-1}(1-\lambda)^i\tilde{z}_{t-i} \quad (1-\lambda)^t W_{\tilde{z},0} \end{aligned}$$

where the sequence $\{\tilde{z}_t\}$ consists of AR(2) observations. In Appendix A we provide details of the derivation of the variance of $W_{\tilde{z},t}$. If t is sufficiently large, the variance of $W_{\tilde{z},t}$ is given by

$$\sigma^2_{W_{\tilde{z},t}} = \left(\frac{\lambda}{2-\lambda}\right) \frac{\phi_1(1 \quad \phi_2)(\lambda-1) \quad (\phi_2-1)(1 \quad \phi_2(\lambda-1)^2)}{(1 \quad \phi_1-\phi_2)(1 \quad \phi_2)(1-\phi_1-\phi_2)(-1 \quad \phi_1(1-\lambda) \quad \phi_2(\lambda-1)^2)}\sigma_a^2 \tag{6}$$

Hence, the control limits for the EWMA control chart for AR(2) data are given by

$$\begin{aligned} UCL &= \mu \quad c\sigma_{W_{\tilde{z},t}} \\ LCL &= \mu - c\sigma_{W_{\tilde{z},t}} \end{aligned}$$

where c is a constant to be chosen by the designer of the control chart; see below. Furthermore, we have to choose λ and provide estimates for ϕ_1, ϕ_2, μ, and σ_a^2. The estimator for μ is the sample mean. The estimation of the parameters of the AR(2) process ϕ_1, ϕ_2, and σ_a^2 can be done by maximum likelihood estimation.

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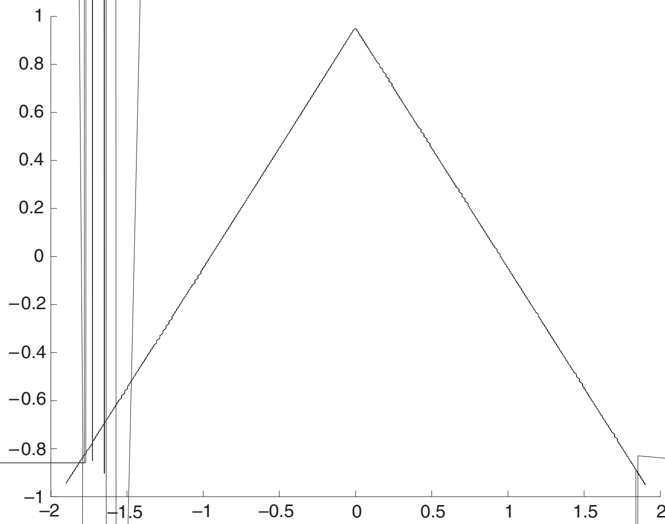
In Appendix B we provide additional properties of the variance in (6). Henceforth, we will refer to the EWMA control charts with the variance in (6) as the modified EWMA control chart. Note that expression (6) is also derived in Fuller[36] using the expression of Zhang[20]. In this article, however, we do not already assume that t is large and we obtain an expression in Appendix A dependent on t.

The designer of the control chart has to choose the average run length in the in-control situation (ARL(0)) and the magnitude of the shift (δ) in the mean that it is desired to detect. Based on ARL(0), δ, and for given ϕ_1 and ϕ_2, the optimal λ and c can be obtained. Crowder[37] expressed the ARL function as an integral equation (i.e. a Fredholm integral of the second kind), which he solved with numerical methods. For that he fixed $\text{ARL}(0)=370$ and calculated the appropriate c for given λ in the IID case. As an alternative approach Crowder[37] suggested to interpolate the results from his Table 1. VanBrackle and Reynolds[33] and Wieringa[34] noted that finding the numerical solution from the integral equation of the ARL function for an AR(1) process is computationally taxing. Both carried out extensive simulations to find the ARL in different situations. This strategy is also followed by Zhang[20,21] and Schmid[32].

The present article would be useful to the practitioner if we add some method for determining c for a menu of values of ϕ_1, ϕ_2 , ARL(0), and λ. However, generating a useful table, i.e. enough combinations of ϕ_1s, ϕ_2s, λs, and ARL(0)s, would require extensive computations. Instead, we have produced a simple polynomial approximation based on published tables by Zhang[20,21] and Crowder[37]. In today's computing environment, with the proliferation of spreadsheet programs such approximations are very convenient for engineers to use. Thus, for a desired combination of ϕ_1, ϕ_2, λ, and ARL(0) we can compute c from

$$\begin{aligned} c = -0.33 - 0.19\phi_1 \quad 0.07\phi_2 \quad 1.72\lambda \quad 0.61\log(\text{ARL}(0)) - 0.29\phi_1^2 - 0.94\lambda^2 \\ -0.02\log(\text{ARL}(0))^2 - 0.35 \end{aligned}$$


1
0.8
0.6
0.4
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−0.2
−0.4
−0.6
−0.8
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−1
−0.5
0
0.5
1
1.5
2

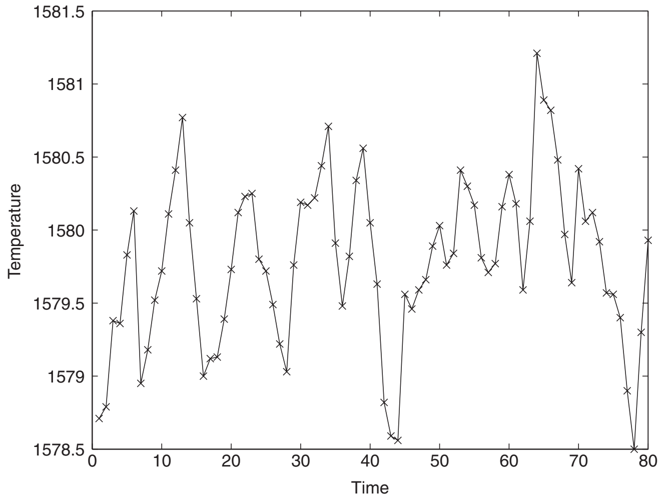

Figure 2. The time series plot of the furnace temperature data

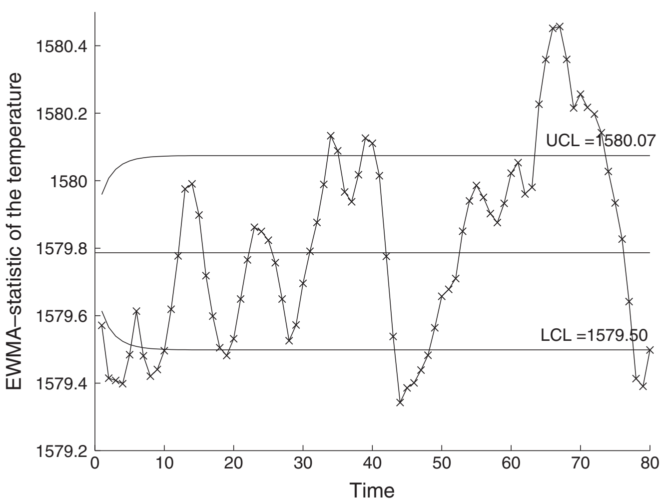

Figure 3. The EWMA control chart of the furnace temperature data

We see that the control limits now are inflated so that the EWMA at any point in time is clearly within the control limits. Thus, these limits can be used to monitor the process going forward, a Phase II application.

To study the sensitivity of the inflation factor, in Figure 5 we have expanded a part of the contour plot in Figure 1. Figure 5 shows a 95% confidence region for the parameter estimates for ϕ_1 and ϕ_2. The confidence region is approximately bounded by the contour on the sum of squares surface, see Box *et al.*[29] (p. 245):

$$\mathrm{SS}(\phi_1, \phi_2) = \mathrm{SS}(\hat{\phi}_1, \hat{\phi}_2)\left[1 \quad \frac{\chi^2_\alpha(k)}{n}\right]$$

where SS is the sum of squares of the residuals given the choice of the parameters ϕ_1 and ϕ_2, $\chi^2_\alpha(k)$ is the significant point exceeded by a proportion α of the chi-squared distribution with k degrees of freedom, and n is the number of observations. For the present example, $\mathrm{SS}(\hat{\phi}_1, \hat{\phi}_2) = 10.5$, $\chi^2_{0.05}(2) = 5.99$, and $n = 78$;

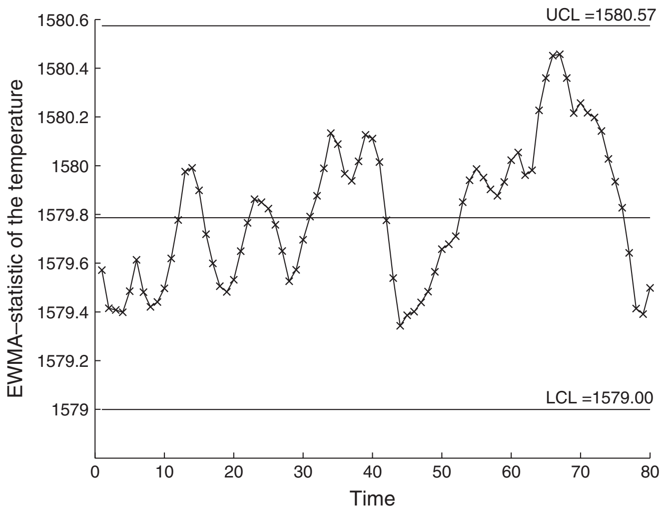

Figure 4. The modified EWMA control chart of the furnace temperature data

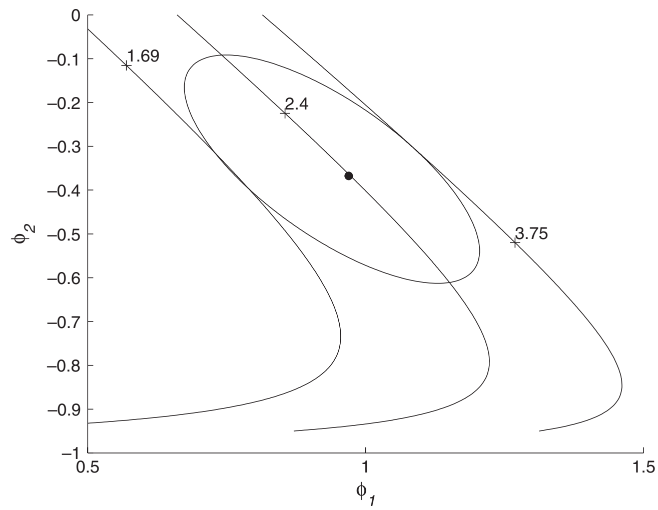

Figure 5. A part of the contour plot of the square root of the inflation factor with the 95% confidence region of the parameters ϕ_1 and ϕ_2

hence $\mathrm{SS}(\phi_1, \phi_2) = 11.3$. Note that $n = 78$ and not 80, because for the first two observations there are no residuals.

As can be seen from this confidence region, the square root of the estimated inflation factor ranges from 1.69 to 3.75. Hence, the number of false alarms using the IF decreases dramatically, but keeps the risk low so that a special cause is not observed.

Note that if the roots are outside the unit circle, the process is non-stationary. Consequently, the variance of the AR(2) process and the EWMA statistic will be infinite. In such cases, process control is not suitable. An alternative would be to use feedback control; see, e.g. Box and Luceño[18].

Further note that the special case of an AR(1) process is treated in Schmid[32], VanBrackle and Reynolds[33], and Wieringa[34]. In particular, note that the formulas of the variance of $W_{\tilde{z},t}$ in (A1) coincide with the formula of the variance of $W_{\tilde{z},t}$ for an AR(1) model when we set $\phi_2 = 0$ and coincide with the formulas of the variance of $W_{\tilde{z},t}$ for the IID model when we set $\phi_1 = \phi_2 = 0$; see Appendix B.

CONCLUSIONS

Autocorrelated observations are common in industry, especially when data are sampled at a high frequency from processes with inertia or carry-over effects. The classical EWMA control chart is non-robust to serial correlation. However, the EWMA control chart is often used even when processes are autocorrelated. As the EWMA control chart is popular, having many desirable properties and widely available in software packages, it is desirable to modify the EWMA control chart to accommodate for commonly encountered situations. Although not covering all situations, the AR(1) and the AR(2) processes cover a relativity wide range of situations encountered in practice. In this article we have provided tools and methods for modifying the EWMA control charts to accommodate for such autocorrelated processes. The approach is based on time series modeling of the process but using the standard EWMA control charts with appropriately modified control limits. This approach is not always optimal, but has the appeal of being simple to introduce to engineers already familiar with standard SPC tools.

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APPENDIX A: DERIVATION OF THE VARIANCE OF $W_{\tilde{z},t}$

From Box *et al.*[29] (p. 27) it follows that

$$\mathrm{Var}(W_{\tilde{z},t}) = \lambda^2 \sum_{i=0}^{t-1}\sum_{j=0}^{t-1}(1-\lambda)^{i\quad j}\gamma_{|j-i|}$$

$$= 2\lambda^2 \sum_{i=0}^{t-2}\sum_{j=i\quad 1}^{t-1}(1-\lambda)^{i\quad j}\gamma_{j-i} \quad \lambda^2\sum_{i=1}^{t-1}(1-\lambda)^{2i}\gamma_0$$

$$= 2\lambda^2 \sum_{i=0}^{t-2}\sum_{j=i\quad 1}^{t-1}(1-\lambda)^{i\quad j}\gamma_{j-i} \quad \gamma_0\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2t}]$$

In Zhang[20] and Schmid[32], it is shown that this expression can be rewritten as

$$\mathrm{Var}(W_{\tilde{z},t}) = \left(\frac{\lambda}{2-\lambda}\right)[\Gamma_t \quad E_t]$$

where

$$\Gamma_t = 2\sum_{k=1}^{t-1}\gamma_k(1-\lambda)^k(1-(1-\lambda)^{2(t-k)}) = 2\left[\sum_{k=1}^{t-1}\gamma_k(1-\lambda)^k - \sum_{k=1}^{t-1}\gamma_k(1-\lambda)^{2t-k}\right]$$

$$E_t = \gamma_0(1-(1-\lambda)^{2t})$$

By substituting the covariance from (3) into Γ_t and after some tedious but simple algebra, we obtain

$$\Gamma_t = 2(v_1 - v_2)^{-1}\left\{\sum_{k=1}^{t-1}(1-\lambda)^k v_1^k \gamma_1 - \sum_{k=1}^{t-1}(1-\lambda)^k v_1^k v_2 \gamma_0\right.$$

$$-\sum_{k=1}^{t-1}(1-\lambda)^k v_2^k \gamma_1 \quad \sum_{k=1}^{t-1}(1-\lambda)^k v_2^k v_1 \gamma_0 - \sum_{k=1}^{t-1}(1-\lambda)^{2t-k} v_1^k \gamma_1$$

$$\left.\sum_{k=1}^{t-1}(1-\lambda)^{2t-k} v_1^k v_2 \gamma_0 \quad \sum_{k=1}^{t-1}(1-\lambda)^{2t-k} v_2^k \gamma_1 - \sum_{k=1}^{t-1}(1-\lambda)^{2t-k} v_2^k v_1 \gamma_0\right\}$$

$$= 2(v_1 - v_2)^{-1}\{(\gamma_1 - v_2\gamma_0) f(v_1) - (\gamma_1 - v_1\gamma_0) f(v_2) - (\gamma_1 - v_2\gamma_0) g(v_1) \quad (\gamma_1 - v_1\gamma_0) g(v_2)\}$$

where $f(x) = \sum_{k=1}^{t-1}(1-\lambda)^k x^k$ and $g(x) = \sum_{k=1}^{t-1}(1-\lambda)^{2t-k} x^k$. Now using the fact that $\sum_{k=1}^{n} r^k = r(1-r^n)/(1-r)$, $f(x)$ and $g(x)$ can be rewritten as

$$f(x) = \frac{(1-\lambda)^t x^t - (1-\lambda)x}{(1-\lambda)x - 1}$$

and

$$g(x) = \frac{(1-\lambda)^{2t} x - (1-\lambda)^{t \quad 1} x^t}{1 - \lambda - x}$$

Substituting these expression into Γ_t, we obtain after some straightforward simplifications that

$$\Gamma_t = 2(v_1 - v_2)^{-1}\{\lambda(\lambda - 2)(A_t - B_t) \quad C_t \quad D\}$$

where

$$A_t = \frac{v_2^{t \quad 1}(1-\lambda)^t (\gamma_1 - v_1\gamma_0)}{(\lambda \quad v_2 - 1)((1-\lambda)v_2 - 1)}$$

$$B_t = \frac{v_1^t (1-\lambda)^t (\gamma_1 - v_2\gamma_0)}{(\lambda \quad v_1 - 1)((1-\lambda)v_1 - 1)}$$

$$C_t = (1-\lambda)^{2t \quad 1}(v_1 - v_2)\left(\frac{-\gamma_1 \quad \gamma_0 v_1 v_2 (1-\lambda)^{-1}}{(\lambda \quad v_1 - 1)(\lambda \quad v_2 - 1)}\right)$$

$$D = (1-\lambda)(v_1 - v_2)\left(\frac{\gamma_1 - \gamma_0 v_1 v_2 (1-\lambda)}{((1-\lambda)(v_1 - 1))((1-\lambda)v_2 - 1)}\right)$$

From this it follows that

$$\mathrm{Var}(W_{\bar{z},t}) = \left(\frac{\lambda}{2-\lambda}\right)[\Gamma_t \quad E_t]$$

$$= \left(\frac{\lambda}{2-\lambda}\right)[2(v_1 - v_2)^{-1}(\lambda(\lambda - 2)(A_t - B_t) \quad C_t \quad D) \quad E_t]$$

Now letting $t \to \infty$, it can be seen that A_t, B_t, and C_t go to zero. We are then left with

$$\begin{aligned}\mathrm{Var}(W_{\tilde{z},t}) &= \frac{2\lambda(\lambda-1)(\gamma_1 \quad \gamma_0 v_1 v_2(\lambda-1))}{(\lambda-2)(1 \quad v_1(\lambda-1))(1 \quad v_2(\lambda-1))} \quad \frac{\gamma_0\lambda}{2-\lambda} \\ &= \left(\frac{\lambda}{2-\lambda}\right)\left(\frac{2(1-\lambda)(\gamma_1 \quad \gamma_0 v_1 v_2(\lambda-1))}{(1 \quad v_1(\lambda-1))(1 \quad v_2(\lambda-1))} \quad \gamma_0\right)\end{aligned}$$

Finally substituting the expressions for γ_0 and γ_1 into (1) and (2) and v_1 and v_2 from (4) and (5), we obtain the desired explicit asymptotic expression for the variance:

$$\mathrm{Var}(W_{\tilde{z},t}) = \left(\frac{\lambda}{2-\lambda}\right) \frac{\phi_1(1 \quad \phi_2)(\lambda-1) \quad (\phi_2-1)(1 \quad \phi_2(\lambda-1)^2)}{(1 \quad \phi_1-\phi_2)(1 \quad \phi_2)(1-\phi_1-\phi_2)(-1 \quad \phi_1(1-\lambda) \quad \phi_2(\lambda-1)^2)}\sigma_a^2 \tag{A1}$$

APPENDIX B: PROPERTIES OF THE VARIANCE OF $W_{\tilde{z},t}$

Here, we provide several properties of the variance of the EWMA statistic (6) or equivalently (A1).

Property 1. *If* $\phi_2 = 0$ *we have an AR*(1) *process. The variance of the EWMA statistic*, $W_{\tilde{z},t}$, *given by* (A1) *for large* t *reduces to*

$$\mathrm{Var}(W_{\tilde{z},t}) = \left(\frac{\lambda}{2-\lambda}\right) \frac{1 \quad \phi_1(1-\lambda)}{(1-\phi_1^2)(1-\phi_1(1-\lambda))}\sigma_a^2$$

Schmid[32], *VanBrackle and Reynolds*[33], *and Wieringa*[34] *also found this result.*

Property 2. *If* $\phi_1 = \phi_2 = 0$, *the AR*(2) *process reduces to an independent and identically distributed random noise process*, *and the variance of the EWMA statistic* $W_{\tilde{z},t}$ *given by* (A1) *for large* t *reduces to the well-known standard expression*

$$\mathrm{Var}(W_{\tilde{z},t}) = \mathrm{Var}(W_{X,t}) = \left(\frac{\lambda}{2-\lambda}\right)\sigma_a^2$$

Property 3. *If* $\lambda = 1$ *in the expression for the EWMA statistic*, *then the EWMA statistic reduces to* $\tilde{z}_t$, *simply the AR*(2) *process itself. In other words*, *the EWMA has no memory and the variance is the same as the process variance*

$$\mathrm{Var}(W_{\tilde{z},t}) = \left(\frac{1-\phi_2}{1 \quad \phi_2}\right) \frac{\sigma_a^2}{(1-\phi_2)^2-\phi_1^2}$$

This result can also be found in Box et al.[29] (p. 62).

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